Johnson Quantile-Parameterized Distributions

Christopher C. Hadlock and J. Eric Bickel
Graduate Program in Operations Research & Industrial Engineering
The University of Texas, Austin, Texas 78712
cchadlock@utexas.edu, ebickel@utexas.edu

It is common decision analysis practice to elicit quantiles of continuous uncertainties and then fit a continuous probability distribution to the corresponding probability-quantile pairs. This process is inconvenient because it requires curve-fitting and the best-fit distribution will often not honor the assessed points. By strategically extending the Johnson Distribution System, we develop a new distribution system that honors any symmetric percentile triplet of quantile assessments (e.g. the 10th-50th-90th) in conjunction with known support bounds. Further, our new system is directly parameterized by the assessed quantiles and support bounds, eliminating the need to apply a fitting procedure. Our new system is practical, flexible, and, as we demonstrate, able to match the shapes of numerous commonly-named distributions.

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1. Introduction

Suppose X is a continuous random variable with cumulative distribution function (cdf)

$$p = F_X(x) \equiv P(X \le x) , \tag{1}$$

and quantile function

$$x_p = Q_X(p) \equiv \inf\{x \in \mathbb{R} : p \le F(x)\} . \tag{2}$$

If F_X is continuous and increasing over the support of X, which we assume here, then Q_X is simply the inverse cdf of X. We refer to x_p as the p-level quantile of X or the ($p \cdot 100$)-th percentile of X. For example, $x_{0.5} = Q_X(0.5)$ denotes the 0.5-level quantile, or 50th percentile (P50) of X.

In many decision analysis applications, analysts assess uncertainty by eliciting a limited number (three is common) of (p, x_p) pairs from a subject-matter expert. For example, it is common to assess the 0.10-, 0.50-, and 0.90-level quantiles or, equivalently, the 10th, 50th, and 90th percentiles[1]. We assume that the (p, x_p) pairs are *coherent* – they satisfy the axioms of probability. Given a set of assessments, analysts may then fit a continuous cdf to these points. In practice, this process may require solving a non-linear optimization problem for the distribution parameters, which some analysts may find difficult to

[1] For example, see McNamee and Celona (1990), Hammond and Bickel (2013a), and Hurst et al. (2000).

implement. More importantly, however, the best-fit cdf will often not honor the assessed probability-quantile pairs. For example, if the best-fit distribution family is specified by two parameters, such as the mean and variance, then it is likely that the selected distribution will not pass through any points provided by the expert. This can cause confusion and decrease trust in the analysis.

Recently, Keelin and Powley (2011) developed the idea of quantile-parameterized distributions (QPDs). QPDs are parameterized by, and thus precisely honor, the (p, x_p) pairs elicited from an expert. Specific to this new class of QPDs is Keelin and Powley's *simple Q-normal* (SQN) system. Distributions within the SQN system are parameterized by four distinct (p, x_p) assessment pairs; for example, the 25th, 50th, 75th, and 90th percentiles. The flexibility of the SQN system is limited, however, since it cannot honor *any* four coherent (p, x_p) pairs.

In this paper, we develop a new family of continuous distributions that are parameterized by their quantiles, with bounded and semi-bounded support[2]. Our system is an extension of the Johnson distribution system (JDS), but can honor any *symmetric percentile triplet* (which we formally define in §2), making it much more flexible than the SQN, including its extensions by Powley (2013) – which we discuss later. We refer to our new family of probability distributions as the J-QPD system. We also show that our J-QPD system can closely approximate a vast array of commonly-named distributions (e.g., beta, gamma, lognormal, etc.). While our system is new, we must stress that it is not unique. Indeed, there are an infinite number of distributions that pass through any finite set of probability-quantile pairs. As we explain more fully below, our objective is to develop a family of continuous distributions that honor assessed quantiles, are straight-forward to implement in practice, and are close to named distributions exhibiting the same probability-quantile pairs.

The remainder of this paper is organized as follows. §2 specifies five criteria that guide the development of our new distribution family. In §3, we analyze two existing distribution systems that are amenable to quantile-parameterized representations, along with their respective strengths and limitations. In §4, we extend the JDS to design our new J-QPD distribution system. In §5, we quantify the ability of

[2] We focus on the bounded and semi-bounded cases for developing our new system on the presumption that many physical quantities have a finite and known *lower* (but not necessarily upper) limit of support (e.g., oil reserves cannot be negative). However we also generate several unbounded distributions in §6, wherein we identify several limiting distributions.

the J-QPD system to closely match the shape of common distribution families. In §6, we examine the flexibility of the J-QPD system, and identify several limiting distributions. In §7, we illustrate how to implement the J-QPD distribution system. Finally, we conclude in §8, noting the benefits and limitations of our new distribution system, including important advantages and disadvantages of J-QPD compared to the QPDs of Keelin and Powley (2011), Powley (2013), and Keelin (2016). We also suggest several extensions for future work.

2. Desirable Features of a Distribution Family

Given the wide array of potential continuous distribution families from which one may choose, it is helpful to have some criteria or desiderata that the distribution family should meet, if possible. In this section, we propose a set of criteria that seem, to us, desirable from the perspective of decision analysis practice. We begin with some definitions that will make our development more efficient.

2.1 Definitions

In the context of elicitation, when assessing an uncertainty from a subject-matter expert, a common practice is to elicit a triplet of low-base-high quantile values of the form: $\mathbf{x}_\alpha = (x_\alpha, x_{0.50}, x_{1-\alpha})$. For example, $\mathbf{x}_{0.1} = (x_{0.10}, x_{0.50}, x_{0.90})$ denotes the vector of 10th, 50th, and 90th percentiles. To streamline our discussion, we introduce the following definitions:

Definition 1. Consider any $\alpha \in (0,0.50)$, and define an ***α*-level symmetric percentile triplet (*α*-SPT)** as a vector $\mathbf{x}_\alpha = (x_\alpha, x_{0.50}, x_{1-\alpha})$, where x_α denotes the *α*-level quantile for the random variable *X*.

Definition 2. Presume that the lower, *l*, and upper, *u*, support bounds ($l < u$) of *X* are specified and that an expert provides $\mathbf{x}_\alpha = (x_\alpha, x_{0.50}, x_{1-\alpha})$ for some $\alpha \in (0,0.50)$. Collectively, define $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u) = (l, x_\alpha, x_{0.50}, x_{1-\alpha}, u)$.

Definition 3. The vector $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$ is **compatible** if and only if $\alpha \in (0,0.50)$ and $l < x_\alpha < x_{0.50} < x_{1-\alpha} < u$.

Definition 4. Define $Q(p;\boldsymbol{\theta}_\alpha)$ as a quantile function (QF) on $p \in [0,1]$ for some probability distribution, and having distribution parameters given by $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$.

2.2 Desiderata

We seek a probability distribution system, defined by $Q(p;\boldsymbol{\theta}_\alpha)$, that satisfies five criteria:

(1) *Quantile-Parameterized* – The distribution is characterized by $Q(p;\boldsymbol{\theta}_\alpha)$, which itself is directly parameterized by $\boldsymbol{\theta}_\alpha$, the assessed quantiles and desired support bounds. This has several benefits. First, as discussed above, this is convenient because it eliminates the need to apply a fit procedure (e.g., solve an optimization problem). Second, $Q(p;\boldsymbol{\theta}_\alpha)$ will honor $\boldsymbol{\theta}_\alpha$, the assessed quantiles and known limits of support. Third, having $Q(p;\boldsymbol{\theta}_\alpha)$ allows the analyst to directly implement Monte Carlo sampling, via the inverse transform method. Fourth, having $Q(p;\boldsymbol{\theta}_\alpha)$ allows for the computation of additional quantiles, allowing the analyst to verify the assessment with an expert by checking additional points. Finally, possession of $Q(p;\boldsymbol{\theta}_\alpha)$ facilitates the subsequent construction of discrete approximations; see Bickel et al. (2011) and Hammond and Bickel (2013a, 2013b), for a review of discretization methods, along with recent extensions.

(2) *Availability of CDF* – $Q(p;\boldsymbol{\theta}_\alpha)$ is directly invertible, so that the distribution cdf, denoted $F(x;\boldsymbol{\theta}_\alpha)$, is readily available. This allows the analyst to verify the assessment with an expert by checking additional points, similar to $Q(p;\boldsymbol{\theta}_\alpha)$. Also, density functions (pdfs) can readily be obtained from $F(x;\boldsymbol{\theta}_\alpha)$ via differentiation.

(3) *Maximally-Feasible* – For *any* compatible $\boldsymbol{\theta}_\alpha$, the QF given by $Q(p;\boldsymbol{\theta}_\alpha)$ satisfies: $Q(0;\boldsymbol{\theta}_\alpha)=l$, $Q(\alpha;\boldsymbol{\theta}_\alpha)=x_\alpha$, $Q(0.50;\boldsymbol{\theta}_\alpha)=x_{0.5}$, $Q(1-\alpha;\boldsymbol{\theta}_\alpha)=x_{1-\alpha}$, and $Q(1;\boldsymbol{\theta}_\alpha)=u$. That is, the distribution characterized by $Q(p;\boldsymbol{\theta}_\alpha)$ honors both the support bounds, and the assessed quantiles given by $\mathbf{x}_\alpha$, for *any* compatible $\boldsymbol{\theta}_\alpha$. We refer to this as the **maximally-feasible (MF) property**, within the context of our setup.[3]

(4) *Closeness to Commonly-Named Distributions* – $Q(p;\boldsymbol{\theta}_\alpha)$ closely[4] approximates the QF of numerous commonly-named distributions that share the same $\boldsymbol{\theta}_\alpha$; i.e., the same *α*-SPT and support. In the case of finite support, we would like the distribution family to closely approximate the ∩- (bell-), J-, and U-shaped distributions contained in the beta family. For semi-infinite support, we would like the distribution family to closely approximate the shapes of the lognormal, gamma, inverse-gamma, and beta-prime distributions.

[3] In this context, maximally-feasible (MF) does not necessarily guarantee that there exists a J-QPD distribution that satisfies a set of five coherent (p, x_p) pairs that are *not* of the SPT structure defined in Definitions 1 and 2; for example, the {0th, 25th, 50th, 95th, 100th} percentiles, which [collectively] are not of SPT form for some $\alpha \in (0,0.50)$.
[4] We more precisely define our notion of "closeness" in §5. Also

(5) *Highly Flexible* – Specifically, we refer to the span of a system within the skewness-kurtosis space developed by Pearson (1895, 1901, 1916).

The "Closeness" desideratum is also motivated by the context in which knowledge of smoothness is present, which we believe to be quite reasonable in practice. In addition, we seek a distribution capable of representing phenomena whose underlying distribution is derived from an underlying physical process. Examples include the normal (lognormal) distributions, which approximately occur in the summation (product) of independent or weakly-dependent random variables due to Central Limit Theorem (CLT) effects, and the exponential distributions for waiting times.

2.3 Illustrative Examples

Throughout this paper, we rely on two illustrative examples (bounded and semi-bounded support) to demonstrate our new distribution system. In the case of bounded support, suppose an expert has been asked to assess peak market share for a new product and provides $\boldsymbol{\theta}_{0.1} = (0, 0.32, 0.40, 0.60, 1)$. Figure 1 presents the best fit beta distribution QF (dashed line), using least-squares, subject to honoring the bounds, and the QF for our new J-QPD family (solid line). Notice that there is no beta distribution satisfying all five points in $\boldsymbol{\theta}_{0.10}$, since generalized beta distributions are specified by four points, and thus the best we can do is fit a distribution through these five given points. On the other hand, J-QPD exactly matches the provided assessments. Again, we do not claim that J-QPD is the only distribution that passes through these five points.

Figure 1. Beta Least-Squares Fit versus a J-QPD Assignment for $\boldsymbol{\theta}_{0.10} = (0, 0.32, 0.40, 0.60, 1)$

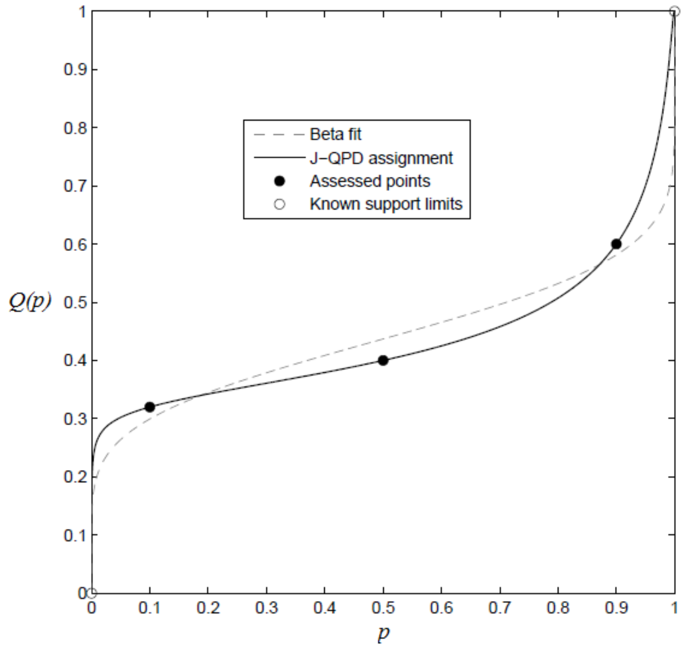

Figure 2 presents an example with semi-bounded support. In this case, an expert is assessing the uncertainty surrounding the capital expenditures (CAPEX) of a drilling venture and provides $\boldsymbol{\theta}_{0.10} = (0, 30, 40, 60, \infty)$ \$MM.[5] Figure 2 shows least-squares fits for Weibull and lognormal QFs (dashed lines) with respect to $\boldsymbol{\theta}_{0.10} = (0, 30, 40, 60, \infty)$, along with our J-QPD assignment (solid line). In this case, there is no distribution within the Weibull or lognormal families that honors $\boldsymbol{\theta}_{0.10}$. In addition, while not shown, no gamma or beta-prime distribution honors $\boldsymbol{\theta}_{0.10}$ either.

In the market share (CAPEX) example, the least-squares fit entails solving a non-linear optimization problem over the shape parameters of the beta (lognormal, Weibull) distribution(s), to minimize the mean-squared error. More importantly, however, the commonly-named distributions selected for the fit in each case do not honor the points in $\boldsymbol{\theta}_{0.10}$. Thus, we see that the beta, Weibull, lognormal, and gamma distributions fail to satisfy Criteria 1 (quantile-parameterized) and 3 (maximally-feasible). More generally, distributions within the flexible family developed by Pearson (1895, 1901, 1916) also fail to satisfy Criteria 1 and 3, including: beta, beta-prime, gamma, inverse-gamma, and Type IV.

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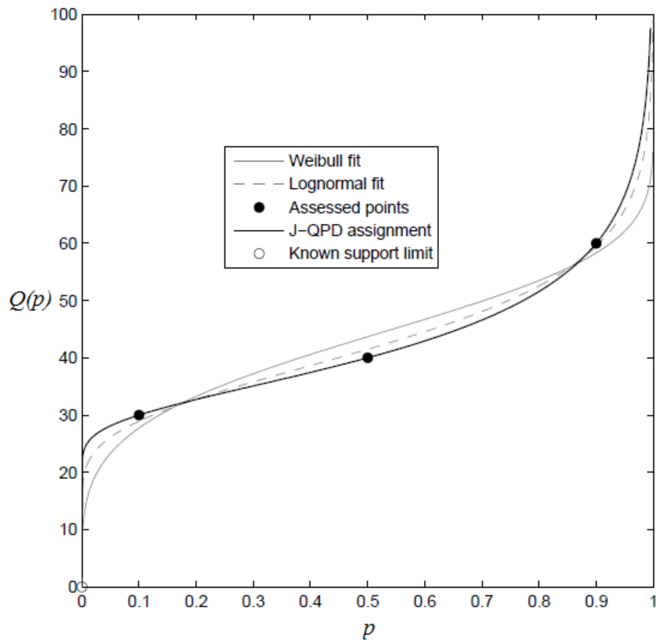
[5] Of course, CAPEX cannot be infinite. Assuming it is unbounded above is a modeling decision meant to represent the fact that the upper bound is unknown and possibly several orders of magnitude larger than the 90th percentile.

Figure 2. Least-Squares Fits versus a J-QPD Assignment for $\boldsymbol{\theta}_{0.10} = (0, 30, 40, 60, \infty)$



There are several recently-proposed distributions that nearly meet these five desiderata. The straight-line approach presented by Clemen and Reilly (1996), as well as maximum-entropy methods proposed by Abbas (2003) seek to add no additional information to an uncertainty other than the assessed quantile-probability pairs, by assigning uniform conditional distributions between adjacent percentile assessments. These methods (and their variants) are maximally-feasible within our construct, and have closed-form pdfs, cdfs, and quantile functions. The same applies to the General Segmented Distributions (GSD) proposed by Vander Wielen and Vander Wielen (2015), among others[6], and to discrete approximations (discretization). However, if knowledge of smoothness is present, which we find to often be the case and assume here, then the unwarranted "kinks" (discontinuous derivatives) inherent in these distributions may less-accurately represent an expert's knowledge, as suggested in Keelin (2016) and Garthwaite et al. (2005).

Furthermore, due to their lumpy nature, these distributions generally fail to satisfy Desiderata 4, which is motivated in part by the desire of our distributions to be able to capture phenomena whose distribution is derived from a well-known underlying physical process – what Keelin (2016) refers to as Type I distributions. Examples include: the normal (lognormal) distributions, which approximately occur

[6] See, for example: Kotz and Van Dorp (2002a, 2002b, 2006).

as the summation (product) of independent or weakly-dependent random variables due to Central Limit Theorem (CLT) effects; exponential distributions for inter-arrival times within a Poisson Process; Weibull distributions, and related extensions[7], in reliability theory for modeling the time between adjacent component failures in complex systems. Unlike straight-line, maximum entropy, GSD, etc., our inherently smooth J-QPD distributions presented in this paper precisely subsume the pervasive normal and lognormal distributions as special cases, but can also approximate Weibull, gamma, beta, and numerous other commonly-named distributions with potent accuracy. Moreover, we show that J-QPD distributions, while smooth, can approximate triangular distributions with reasonable accuracy.

As we show below, our new J-QPD system meets all of our criteria outlined above. The J-QPD system consists of two major subfamilies:

(1) J-QPD-B (bounded): Has finite lower and upper support bounds, [l, u], and is parameterized by any compatible $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$.

(2) J-QPD-S (semi-bounded): Has support on [l, ∞) and is parameterized by any compatible $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, \infty)$.

The market share example of Figure 1 uses a J-QPD-B distribution assignment, while the CAPEX example in Figure 2 uses a J-QPD-S distribution assignment.

3. The Simple Q-Normal and Johnson Distribution Systems

In this section we carefully analyze two flexible distribution systems that are amenable to quantile-parameterized representations:

- The **Simple Q-Normal system (SQN)**, developed by Keelin and Powley (2011), including relevant extensions by Powley (2013).
- The **Johnson distribution system (JDS)**, developed by Johnson (1949).

It is important to introduce the SQN system, since it most closely relates to our new J-QPD system. In addition, the extensions to the SQN noted by Powley (2013) contain many similar features to our new system. However, as we show, the SQN and its extensions possess several shortcomings that limit their

[7] For example, see Mudholkar and Srivastava (1993).

usefulness, compared to our new J-QPD system. We also review the JDS in this section, since it forms the basis for designing the new J-QPD system in §4.

3.1 The Simple Q-Normal System (SQN)

The standard SQN system (Keelin and Powley 2011) is characterized by its QF:

$$Q_{SQN}(p) = a + b \cdot p + c \cdot Q_N(p) + d \cdot p \cdot Q_N(p),$$

where $Q_N \equiv \Phi^{-1}$ is the standard normal quantile function. The parameters, (a, b, c, d), uniquely determine a distribution within the SQN system, via the solution to a linear system. For instance, if the $(x_{0.25}, x_{0.50}, x_{0.75}, x_{0.90})$ quantiles are given, then we can easily obtain (a, b, c, d) by solving:

$$\begin{pmatrix} x_{0.25} \\ x_{0.50} \\ x_{0.75} \\ x_{0.90} \end{pmatrix} = \begin{pmatrix} 1 & 0.25 & \Phi^{-1}(0.25) & 0.25\Phi^{-1}(0.25) \\ 1 & 0.50 & \Phi^{-1}(0.50) & 0.50\Phi^{-1}(0.50) \\ 1 & 0.75 & \Phi^{-1}(0.75) & 0.75\Phi^{-1}(0.75) \\ 1 & 0.90 & \Phi^{-1}(0.90) & 0.90\Phi^{-1}(0.90) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

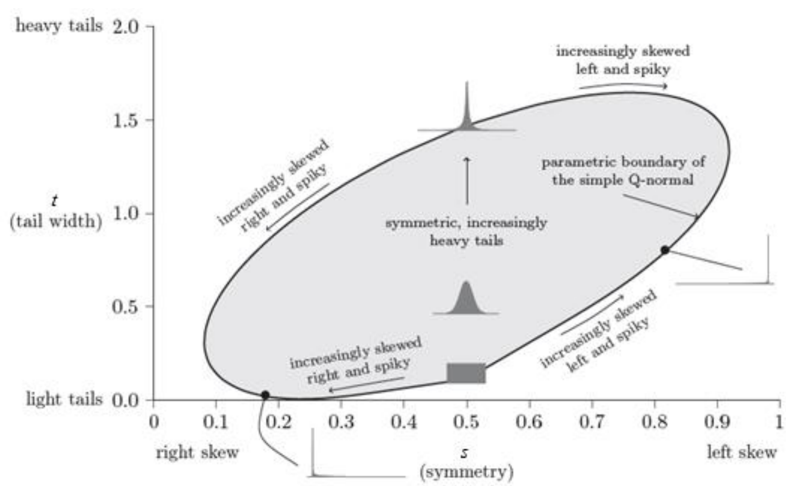
Powley (2013) characterizes the feasibility and flexibility of the SQN system using quantile measures of distribution symmetry (denoted s) and tail-width (denoted t), defined as follows:

$$s = \frac{x_{0.50} - x_{0.10}}{x_{0.90} - x_{0.10}},$$

$$t = \frac{x_{0.10} - x_{0.01}}{x_{0.90} - x_{0.10}}.$$

Figure 3 shows the feasible region of the SQN within the {s, t} space, along with illustrations of the shapes (densities) of several distributions.

Figure 3. Feasible Region of the SQN System in the {*s*, *t*} Space - Powley (2013)



While the SQN is quite tractable, Figure 3 demonstrates that is not possible to assign an SQN distribution to an arbitrary vector of four compatible quantiles. Rather, the quantiles must produce a feasible {*s*, *t*} pair. Points outside of the feasible region in Figure 3 do not correspond to an SQN QF.

In addition, being based on the standard normal distribution, SQN distributions have support on (-∞, ∞). Using Q-transformations (which we introduce in §4) upon the SQN, and eliminating one of the four basis functions, Powley (2013) produces two new distribution systems: one with bounded support; one with semi-bounded support. We refer to the first system as the Probit-SQN (or P-SQN) system, which can be generalized to a location-scale family with bounded support on [*l*, *u*], and having the following QF characterization[8]:

$$Q_{P-SQN}(p) = l + (u-l)\Phi(\xi + (a+bp)\Phi^{-1}(p)). \tag{3}$$

Note that (ξ, a, b) are shape parameters. Given $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$, we can solve for (ξ, a, b), and subsequently re-parameterize equation (3) to satisfy Criterion 1 (quantile-parameterized). The corresponding expressions for (ξ, a, b) are:

$$\xi = \Phi^{-1}\left(\frac{x_{0.50} - l}{u-l}\right),$$

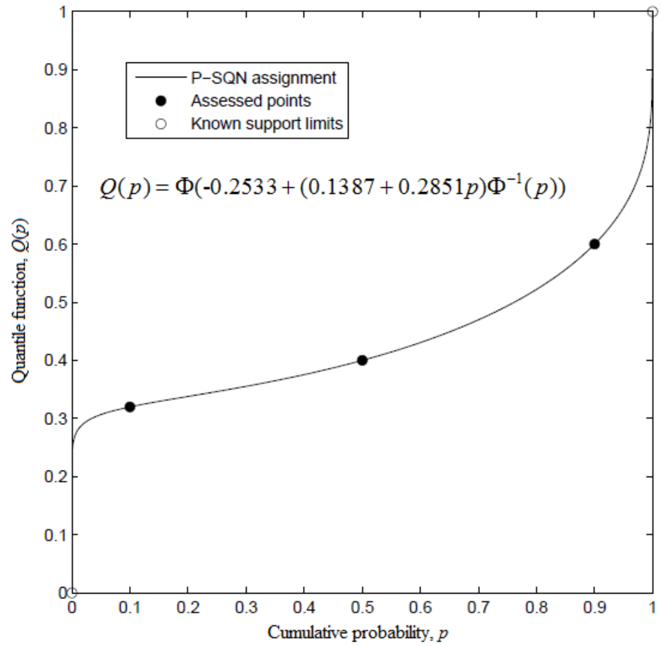
$$b = \frac{\Phi^{-1}\left(\frac{x_\alpha - l}{u-l}\right) + \Phi^{-1}\left(\frac{x_{1-\alpha} - l}{u-l}\right) - 2\xi}{(1-2\alpha)\Phi^{-1}(1-\alpha)},$$

$$a = \frac{\xi - \Phi^{-1}\left(\frac{x_\alpha - l}{u-l}\right) - \alpha b \Phi^{-1}(1-\alpha)}{\Phi^{-1}(1-\alpha)}.$$

We illustrate the usefulness of this re-parameterized form of the P-SQN system by applying it to the market share example from Figure 1. Using the parameter expressions above for (ξ, a, b), given $\boldsymbol{\theta}_{0.10} = (0, 0.32, 0.40, 0.60, 1)$, we obtain the P-SQN QF representation and plot shown in Figure 4.

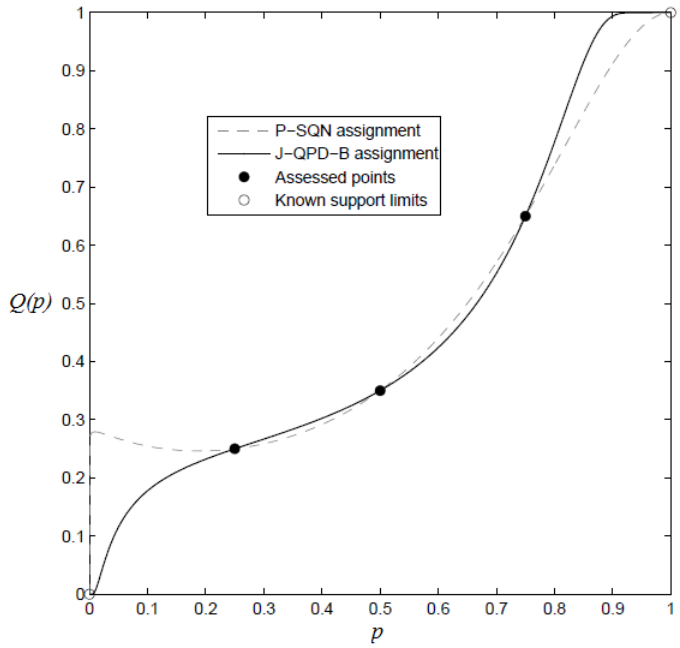
[8] Note that we effectively removed the "*p*" basis function from the standard SQN quantile function, applied the probit operator, Φ, and then implemented changes of location and scale to achieve the arbitrary support bounds given by [*l*, *u*]. The reasoning is as follows. If we apply the probit operator to the standard SQN as is, then all four parameters become shape parameters. Combined with location and scale, the resulting distributions would require six points. However, our SPT setup involves five points (an SPT and the two support bounds). Thus, we eliminate one basis function in order to remove a degree of freedom.

Figure 4. P-SQN Assignment for $\boldsymbol{\theta}_{0.10} = (0, 0.32, 0.40, 0.60, 1)$



Thus, unlike the beta distribution, the P-SQN distribution satisfies Criteria 1 (quantile parameterized). However, the QF representation given in (3) is not invertible, and thus the P-SQN does not satisfy Criterion 2 (availability of the cdf). In addition, the P-SQN distribution does not satisfy Criterion 3 (maximally-feasible). To illustrate, Figure 5 shows the P-SQN QF assignment (dashed) for $\boldsymbol{\theta}_{0.25} = (0, 0.25, 0.35, 0.65, 1)$, along with our corresponding J-QPD-B QF assignment (solid). Notice that while the P-SQN assignment satisfies $\boldsymbol{\theta}_{0.25} = (0, 0.25, 0.35, 0.65, 1)$, it is not a valid QF because it is not non-decreasing, whereas our J-QPD-B assignment *is* a QF. Thus, we say that $\boldsymbol{\theta}_{0.25} = (0, 0.25, 0.35, 0.65, 1)$ is infeasible for the P-SQN system, and that the P-SQN system does not satisfy Criterion 3 (maximally-feasible).

Figure 5. P-SQN (dashed) and J-QPD-B (solid) Assignments for $\boldsymbol{\theta}_{0.25} = (0, 0.25, 0.35, 0.65, 1)$



We refer to Powley's second extended system as the Log-SQN (or L-SQN) system, which has semi-bounded support on [l, ∞), and the following QF characterization:

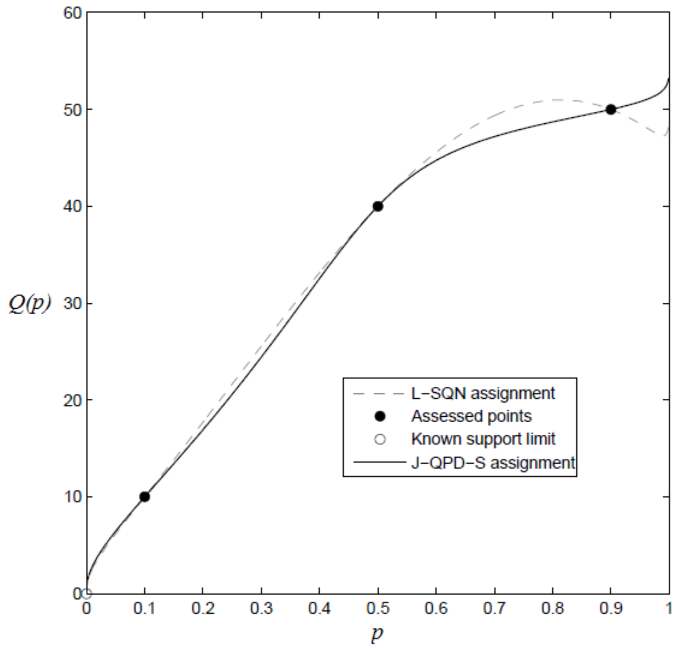
$$Q_{L-SQN}(p) = l + \theta e^{(a+bp)\Phi^{-1}(p)}. \tag{4}$$

In this case, θ is a scale parameter, while a and b are shape parameters. The L-SQN system is constructed in analogous fashion to the P-SQN distribution, except that we apply the "exp" operator instead of the probit operator. Similar to the P-SQN system, we can solve for (θ, a, b) in terms of a given $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, \infty)$. The corresponding expressions for (θ, a, b) are:

$$\theta = x_{0.50},$$
$$b = \frac{\log(x_\alpha - l) + \log(x_{1-\alpha} - l) - 2\log(x_{0.5} - l)}{(1-2\alpha)\Phi^{-1}(1-\alpha)},$$
$$a = \frac{\log(x_{1-\alpha} - l) - \log(x_{0.5} - l) - (1-\alpha)\Phi^{-1}(1-\alpha)b}{\Phi^{-1}(1-\alpha)}.$$

Thus, we see that L-SQN satisfies Criteria 1. However, the QF representation in (4) is also not invertible, and thus L-SQN fails to satisfy Criterion 2. In addition, like P-SQN, the L-SQN distribution fails to satisfy Criterion 3. Figure 6 shows the L-SQN QF assignment for $\boldsymbol{\theta}_{0.1} = (0, 10, 40, 50, \infty)$, along with our corresponding J-QPD-S QF assignment. Note again that the L-SQN assignment is not a valid QF, and thus that $\boldsymbol{\theta}_{0.10} = (0, 10, 40, 50, \infty)$ is infeasible for the L-SQN system.

Figure 6. L-SQN (dashed) and J-QPD-S (solid) Assignments for $\boldsymbol{\theta}_{0.10} = (0, 10, 40, 50, \infty)$



3.2 The Johnson Distribution System (JDS)

The JDS consists of three major subfamilies: the SU, SB, and the lognormal distributions. Since the lognormal distributions have only a single shape parameter, we do not consider them further in this section. The QF for the SU distribution is given by:

$$Q_{SU}(p) = \xi + \lambda \sinh\left(\delta\left(\Phi^{-1}(p) + \gamma\right)\right),\ \delta > 0,\ \lambda > 0. \tag{5}$$

The parameters ξ and λ correspond to location and scale, respectively, while δ and γ are shape parameters. The QF of the standard SU (ξ = 0, λ = 1) results from applying a hyperbolic sine Q-transformation to the QF of a normal distribution. The SU system is quite flexible, but has support on (-∞, ∞) (Johnson 1949). Also, four finite and distinct quantiles are required to specify a particular SU distribution, and this involves solving a non-linear system of equations for the four parameters in (5).

The QF for the SB distribution is given by:

$$Q_{SB}(p) = \xi + \frac{\lambda \exp\left(\delta\left(\Phi^{-1}(p) + \gamma\right)\right)}{1 + \exp\left(\delta\left(\Phi^{-1}(p) + \gamma\right)\right)}, (\delta > 0, \lambda > 0). \tag{6}$$

The parameters ξ and λ correspond to location and scale, whereas δ and γ are shape parameters. The standard SB results from applying the logistic (or inverse-logit) Q-transformation,

$T(y) = \exp(y)/(1+\exp(y))$, to the QF of a normal distribution. SB distributions can be unimodal or bimodal (Johnson 1949)[9].

4. The J-QPD System

In this section we design our new J-QPD system, using the Johnson SU system as a basis for construction, and the five criteria listed in §2 as design specifications. We subsequently show that it satisfies the MF property.

4.1 Engineering the Support of the JDS

One of the most powerful methods for engineering the support of a distribution is by the use of a Q-transformation. The Q-Transformation Rule (QTR), adopted from Gilchrist (2000), is as follows:

The Q-Transformation Rule (QTR) – If $T(x)$ is a non-decreasing function of x, and $Q(p)$ is a QF, then $T(Q(p))$ is a QF.

A corollary of the QTR is that if X is a random variable with QF given by $Q(p)$, then the QF of the transformed variable, $Y=T(X)$, is $T(Q(p))$ (Gilchrist 2000). One well-known Q-transformation used to transform a distribution with support on (-∞, ∞) into a distribution with support on [0,1], is by applying the *inverse-probit Q-transformation* (IP-QT) to its QF. The IP-QT is simply the cdf of the standard normal distribution: $T(x) = \Phi(x)$.

In the next two sections we design our J-QPD distribution system. Our system is similar in spirit to the P- and L-SQN systems. However, as we show, our system has cdf representations (Criterion 2), and is maximally-feasible (Criterion 3). We segment the development of the J-QPD system into bounded (J-QPD-B) and semi-bounded (J-QPD-S) subsystems.

4.2 The J-QPD-B Distributions

Recall from §3.2 that SU distributions have support on (-∞, ∞). To obtain a distribution having arbitrary, but finite, lower and upper support bounds, a natural idea is to apply the well-known IP-QT to the SU quantile function in Equation (5), along with appropriate shifting and scaling to satisfy $\{l, u\}$:

[9] In the development of our new J-QPD system, we also considered the Burr (1973) and Dagum (1977) systems, as well as the Exponentiated Weibull distribution proposed by Mudholkar and Srivastava (1993) (among others), as a basis for construction. However, while all three of these systems have simple analytical expressions for their QF and CDF, and are quite flexible, we cannot directly express their distribution parameters in terms of an SPT and bounds – they are not directly quantile-parameterized (Criterion 1). See footnote 9 for follow-up discussion.

$$Q_1(p) = l + (u-l)\Phi\left(Q_{SU}(p)\right) = l + (u-l)\Phi\left(\xi + \lambda \sinh\left(\delta\left(\Phi^{-1}(p) + \gamma\right)\right)\right).$$

Note that the QTR guarantees that $Q_1(p)$ is a QF, corresponding to a distribution with support on [l, u], so long as we maintain the SU parameter requirements: $\lambda > 0$, $\delta > 0$. Now, we desire for $Q_1(p)$ to be fully parameterized by any compatible $\boldsymbol{\theta}_\alpha$. By inspection, observe that l and u correspond to the 0th and 100th percentiles, respectively, as desired.

With {l, u} presumed known, we currently have four unknowns: {λ, δ, γ, ξ}. However, we can only produce three non-degenerate equations with the low-base-high assessments given in the SPT $\mathbf{x}_\alpha$. A natural idea is to fix one of the parameters, but it is not immediately obvious what constitutes good choices for the fixed parameter and corresponding value(s).

First, note that $Q_1(p)$ is invertible (Criterion 2). Thus, we focus on Criteria 1 (quantile-parameterized) and 3 (maximally-feasible) in making a strategic selection for the choice of the fixed parameter. In particular, the choice of the fixed parameter and value should yield a QF which:

i. Can accommodate any compatible $\boldsymbol{\theta}_\alpha$ (Criterion 3).
ii. Is easy to re-parameterize in terms of $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$ (Criterion 1).

Consider any given real number, $c > 0$. As we soon show, allowing γ to assume one of three possible values, {-c, 0, c}, yields a QF that satisfies (i) for any particular $c > 0$. On the other hand, it turns out that this property does not hold when fixing values for {λ, δ, ξ}.

In choosing a specific value for c, we now bear (ii) in mind. In particular, letting $c = \Phi^{-1}(1-\alpha)$ results in a simple, explicit solution to the distribution parameters in terms of $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$. In particular, for assessments and bounds jointly given in $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u)$, the resulting QF for the J-QPD-B distributions, is as follows:

$$Q_B(p) = l + (u-l)\Phi\left(\xi + \lambda \sinh\left(\delta\left(\Phi^{-1}(p) + nc\right)\right)\right). \tag{7}$$

where,

$$c = \Phi^{-1}(1-\alpha)\,,$$
$$L = \Phi^{-1}\left(\frac{x_\alpha - l}{u-l}\right),\ B = \Phi^{-1}\left(\frac{x_{0.50} - l}{u-l}\right),\ H = \Phi^{-1}\left(\frac{x_{1-\alpha} - l}{u-l}\right),$$
$$n = \operatorname{sgn}(L + H - 2B),$$

$$\xi = \begin{cases} L, \ n = 1 \\ B, \ n = 0 \\ H, \ n = -1 \end{cases}$$

$$\delta = \left(\frac{1}{c}\right)\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right),$$

$$\lambda = \frac{H-L}{\sinh(2\delta c)}.$$

We now note some observations regarding Equation (7):

- $\{\lambda, \delta, \xi\}$ are all specified directly in terms of $\boldsymbol{\theta}_\alpha$, along with constant *c* and parameter *n* (which assumes {-1, 0, 1}, depending on the values of $(l, \mathbf{x}_\alpha, u)$.
- One can easily verify that:
  - $Q_B(0) = l$
  - $Q_B(\alpha) = x_\alpha$
  - $Q_B(0.5) = x_{0.50}$
  - $Q_B(1-\alpha) = x_{1-\alpha}$
  - $Q_B(1) = u$
- Given the simple invertible form of the J-QPD-B QF, we can also produce the cdf (Criterion 2):

$$F_B(x) = \Phi\left(\left(\frac{1}{\delta}\right)\sinh^{-1}\left(\left(\frac{1}{\lambda}\right)\left(\Phi^{-1}\left(\frac{x-l}{u-l}\right) - \xi\right)\right) - nc\right). \tag{8}$$

- Recall that sgn(0) = 0. Examining the expression for *n* in (7) above, we see that this occurs (*n*=0) when: $L + H - 2B = 0$.

$$L + H - 2B = 0 \rightarrow \delta = 0, \lambda = \infty.$$

  Thus, this case violates the parameter requirements ($\delta > 0$) as is. However, using the parameter expressions in (7), we note the following:

$$\lim_{y\to 0}\frac{\sinh(y)}{y} = 1 \rightarrow \lim_{\delta\to 0}\frac{\lambda\sinh\left(\delta\left(\Phi^{-1}(p) + nc\right)\right)}{\lambda\delta\left(\Phi^{-1}(p) + nc\right)} = 1.$$

  However,

$$\lim_{\delta\to 0}\lambda\delta = \lim_{\delta\to 0}\frac{\delta(H-L)}{\sinh(2\delta c)} = \frac{H-L}{2c}.$$

  Therefore, for the special case in which *n* = 0, we define the QF in (7) as follows:

$$Q_B(p) = l + (u-l)\Phi\left(B + \left(\frac{H-L}{2c}\right)\Phi^{-1}(p)\right). \tag{7b}$$

Establishing the MF Property of the J-QPD-B Distribution

Based on the QF representation given in Equation (7), we see thus far that the J-QPD-B distribution satisfies Criteria 1 and 2. We now demonstrate its conformity to Criterion 3 – the MF property.

Proposition 1 (MF Property). Consider any compatible $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u) = (l, x_\alpha, x_{0.50}, x_{1-\alpha}, u)$. There exists a unique quantile function, Q, characterized by (7), that satisfies:

- $Q_B(0) = l$
- $Q_B(\alpha) = x_\alpha$
- $Q_B(0.5) = x_{0.50}$
- $Q_B(1-\alpha) = x_{1-\alpha}$
- $Q_B(1) = u$

Proof. See Appendix A.

Illustrative Example

To illustrate, we now revisit the market share example introduced in §2.3. Given $\boldsymbol{\theta}_{0.10} = (0, 0.32, 0.40, 0.60, 1)$, and using Equation (7), we compute the parameters and construct the corresponding J-QPD-B QF assignment as follows:

$$
\begin{aligned}
&c = \Phi^{-1}(1-\alpha) = \Phi^{-1}(0.90) = 1.2816,\\
&L = \Phi^{-1}\left(\frac{x_\alpha - l}{u-l}\right) = \Phi^{-1}\left(0.32\right) = \text{-}0.4677,\\
&B = \Phi^{-1}\left(\frac{x_{0.50} - l}{u-l}\right) = \Phi^{-1}\left(0.40\right) = \text{-}0.2533,\\
&H = \Phi^{-1}\left(\frac{x_{1-\alpha} - l}{u-l}\right) = \Phi^{-1}\left(0.60\right) = 0.2533,\\
&n = \operatorname{sgn}(L + H - 2B) = \operatorname{sgn}(-0.4677 + 0.2533 - 2(-0.2533)) = 1,\\
&\xi = \begin{cases} L, & n = 1 \\ B, & n = 0 \\ H, & n = -1 \end{cases} = L = -0.4677,\\
&\delta = \left(\frac{1}{c}\right)\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right) = 0.8661,\\
&\lambda = \frac{H-L}{\sinh(2\delta c)} = \frac{0.2533-(-0.4677)}{\sinh(2\cdot 0.8661 \cdot 1.2816)} = 0.1585,\\
&Q_B(p) = l + (u-l)\Phi(\xi + \lambda\sinh(\delta(\Phi^{-1}(p) + nc)))\ .
\end{aligned}
$$

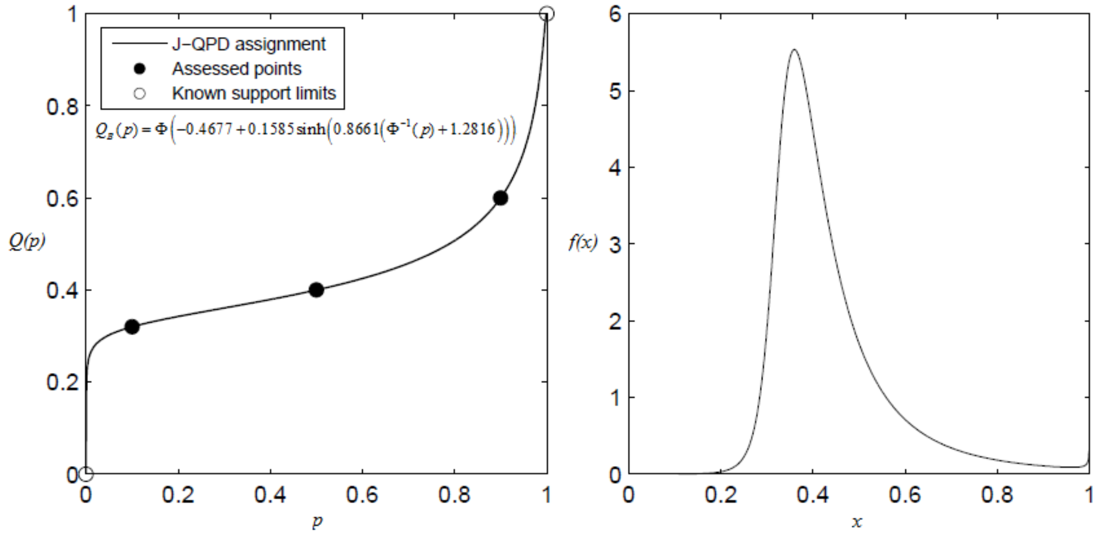
Thus,

$$Q_B(p) = \Phi\Big(-0.4677 + 0.1585\sinh\Big(0.8661\Big(\Phi^{-1}(p) + 1.2816\Big)\Big)\Big).$$

Figure 7 provides a plot of this J-QPD-B QF assignment, along with the corresponding pdf. Using this newly-constructed QF, one can easily confirm that:

$Q_B(0)=0,\ Q_B(0.10)=0.32,\ Q_B(0.50)=0.4,\ Q_B(0.90)=0.6,\ Q_B(1)=1.$

Figure 7. J-QPD-B Assignment for the Market Share Example – QF (left), pdf (right)



Thus far, we have demonstrated that the J-QPD-B distribution satisfies Criteria 1, 2, 3. We defer demonstrating its conformity to Criteria 4 and 5 (closeness to commonly-named distributions, and highly flexible) to §5 and §6, respectively. We now construct the J-QPD-S distribution, which has semi-bounded support.

4.3 The J-QPD-S Distributions

While a finite lower limit of support is sensible for many physical quantities (e.g., non-negativity), experts and/or analysts may not always deem it appropriate to impose a finite value for *u*. Thus, we now develop the J-QPD-S distributions, designed to have support on [*l*, ∞), by once again starting with the SU distributions.

Since SU distributions have support on (-∞, ∞), to obtain a distribution having support on [*l*, ∞) (for some specified *l*), a natural idea is to apply the well-known exponential Q-transformation (Exp) to the SU quantile function, along with appropriate shifting to satisfy *l*:

$$\begin{aligned} Q_1(p) &= l+\exp\left(\xi+\lambda\sinh(\delta(\Phi^{-1}(p)+\gamma))\right) \\ &= l+\theta\exp\left(\lambda\sinh(\delta(\Phi^{-1}(p)+\gamma))\right),\ \delta>0,\ \lambda>0,\ \theta=\exp(\xi). \end{aligned}$$

$Q_1(p)$, as given, has infinite positive moments (see Appendix B for discussion). We thus embed one more strategically-chosen transformation within this QF, along with a re-parameterization analogous to that of the J-QPD-B distributions. Given $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, \infty)$, where l is presumed finite and known, the resulting QF for the J-QPD-S distributions, is as follows:

$$Q_S(p) = l + \theta \exp\left(\lambda \sinh\left(\sinh^{-1}\left(\delta \Phi^{-1}(p)\right) + \sinh^{-1}(nc\delta)\right)\right), \tag{9}$$

where,

$$\begin{aligned}
&c = \Phi^{-1}(1-\alpha), \\
&L = \log(x_\alpha - l),\ B = \log(x_{0.5} - l),\ H = \log(x_{1-\alpha} - l), \\
&n = \operatorname{sgn}(L + H - 2B) \\
&\theta = \begin{cases} x_\alpha - l, & n = 1 \\ x_{0.50} - l, & n = 0 \\ x_{1-\alpha} - l, & n = -1 \end{cases} \\
&\delta = \left(\frac{1}{c}\right)\sinh\left(\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right)\right), \\
&\lambda = \left(\frac{1}{\delta c}\right)\min(H-B, B-L).
\end{aligned}$$

The application of the $\sinh^{-1}$ operator[10] in (9) results in all moments being finite (see Appendix C for a proof). Also, notice that if $L + H - 2B = 0$, then $n = \operatorname{sgn}(L+H-2B) = \operatorname{sgn}(0) = 0$, in which case we have:

$$Q_S(p) = l + \theta \exp\left(\lambda \sinh\left(\sinh^{-1}\left(\delta \Phi^{-1}(p)\right)\right)\right) = l + \theta \exp\left(\lambda \delta \Phi^{-1}(p)\right). \tag{9b}$$

In particular, we precisely recover a lognormal distribution with $\mu = \log(\theta) = \log(x_{0.5} - l)$ and $\sigma = \lambda\delta$ = $(H–B)/c$ (in this case), and shifted to have support on $[l, \infty)$. Thus, the J-QPD-S distributions are a generalization of lognormal distributions, but parameterized by any compatible symmetric percentile triplet and known lower bound (Criterion 1), and effectively having two shape parameters, (λ, δ), whereas lognormal distributions only have the single shape parameter (σ)[11].

[10] We are not the first authors to characterize a distribution quantile function using a combination of the sinh and arcsinh operators, as in (10). See Jones and Pewsey (2009) for the development and application of a “sinh-arcsinh” type transformation upon random variables to generate new probability distributions. To the best of our knowledge, however, our particular combination of sinh and arcsinh applications, along with our strategic re-parameterization given in (10) amounts to a novel probability distribution system parameterized by quantiles.

[11] The Burr, Dagum, and Exponetiated Weibull distributions all have semi-bounded support and two shape parameters, similar to our new J-QPD-S system. However, we find that they are not maximally-feasible. Alternatively, we find that the “sinh” operator affords tremendous flexibility, and combined with the log- and probit

As with the J-QPD-B distributions, we can obtain the cdf (Criterion 2) of the J-QPD-S distribution by simply inverting its QF representation given in (9). This yields:

$$F_S(x) = \Phi\left(\left(\frac{1}{\delta}\right)\sinh\left(\sinh^{-1}\left(\left(\frac{1}{\lambda}\right)\log\left(\frac{x-l}{\theta}\right)\right) - \sinh^{-1}\left(nc\delta\right)\right)\right) \tag{10}$$

Establishing the MF Property of the J-QPD-S Distribution

We now show that the J-QPD-S distribution possesses the MF property (Criterion 3).

Proposition 2 (MF Property). Consider any compatible $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, \infty)$. There exists a unique quantile function, Q, characterized by (9), that satisfies:

- $Q_S(0) = l$
- $Q_S(\alpha) = x_\alpha$
- $Q_S(0.5) = x_{0.50}$
- $Q_S(1-\alpha) = x_{1-\alpha}$
- $Q_S(1) = \infty$

Proof. See Appendix D.

Illustrative Example

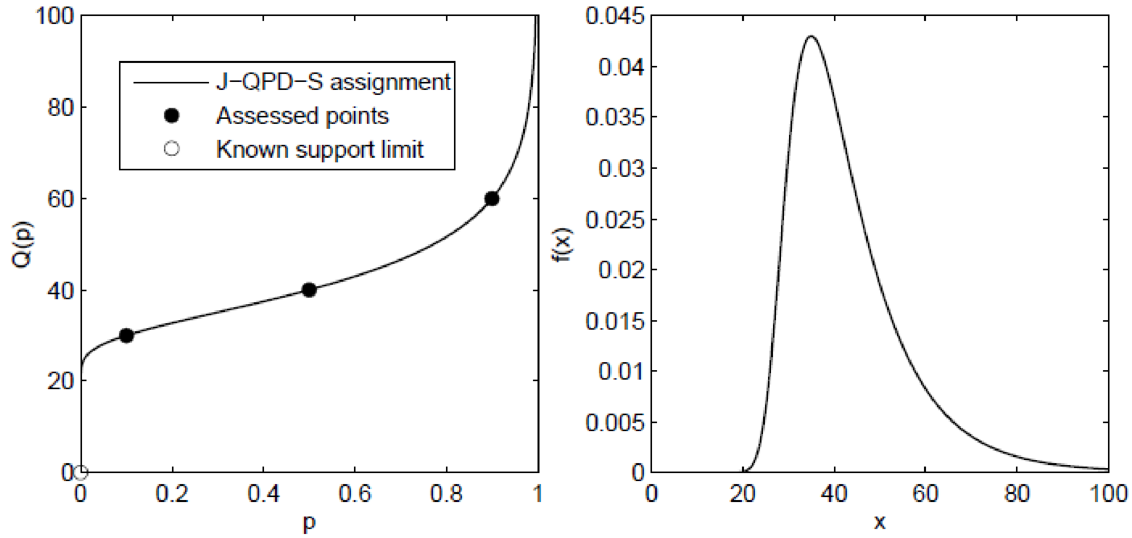
We now apply the J-QPD-S distribution to the CAPEX example introduced in §2.3, in which $\boldsymbol{\theta}_{0.10} = (0, 30, 40, 60, \infty)$ $MM. Using the expressions given in (9), we obtain the following QF assignment:

$$Q_S(p) = 30 \cdot \exp\left(0.4282 \cdot \sinh\left(0.6294 + \sinh^{-1}\left(0.5242 \cdot \Phi^{-1}(p)\right)\right)\right).$$

Figure 8 provides a plot of $Q_S(p)$ for the CAPEX example, along with the corresponding pdf. Similar to the J-QPD-B system in the market share example, notice in Figure 8 that the J-QPD-S assignment precisely honors the low-base-high assessments – in this case, the {10^{th}, 50^{th}, 90^{th}} percentile assessment values of {30, 40, 60} $MM – as well as the known lower limit of support (zero, in this case).

transformations, allows for maximum-feasibility and a diverse range of shapes, including: symmetric and skewed bell-shaped distributions, bimodal (including U-shaped), and J-shaped distributions all within one simple system.

Figure 8. J-QPD-S Assignment for the CAPEX Example – QF (left), pdf (right)

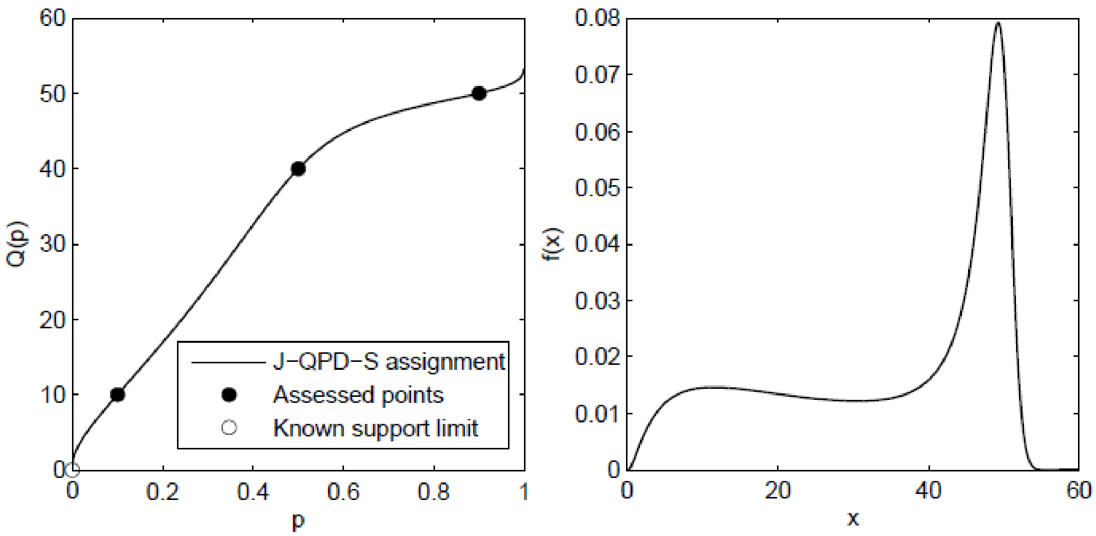


However, recall the more skewed example from Figure 6, where $\boldsymbol{\theta}_{0.10} = (0, 10, 40, 50, \infty)$, and where the L-SQN distribution results in infeasibility. If we apply the J-QPD-S distribution to these four points, we obtain the following QF:

$$Q_S(p) = 50 \cdot \exp\left(0.0644 \cdot \sinh\left(-1.956 + \sinh^{-1}\left(2.7036 \cdot \Phi^{-1}(p)\right)\right)\right).$$

Figure 9 provides a plot of the QF, along with a corresponding pdf. The odd shape is due to the unusual percentile spacing in this example. However, the point is to illustrate the MF property, by showing that the J-QPD-S distribution produces a valid QF exactly honoring $\boldsymbol{\theta}_\alpha$ – where the L-SQN fails, as we saw in Figure 6.

Figure 9. J-QPD-S Assignment for $\boldsymbol{\theta}_{0.10} = (0, 10, 40, 50, \infty)$ – QF (left), pdf (right)



Thus far, we have demonstrated that the J-QPD-S distribution satisfies Criteria 1 through 3. In §5 (6), we demonstrate its conformity to Criterion 4 – closeness to commonly-named distributions (Criterion 6 – flexibility in accordance with Pearson).

4.4 Depicting the Feasibility of the J-QPD Distributions

Having developed the new J-QPD-B and J-QPD-S distribution systems, we now compare the extent of their feasibility to that of several commonly-named distributions. Recall that in both cases, we have a given finite lower bound (*l*), along with an elicited SPT, $\mathbf{x}_\alpha$. Without loss of generality, we normalize the J-QPD distributions in order to remove location and scale. In particular, fix any $\alpha \in (0, 0.50)$, and consider the following normalizing measures:

$$s_\alpha = \frac{x_{0.50} - l}{x_{1-\alpha} - l}$$

$$t_\alpha = \frac{x_\alpha - l}{x_{0.50} - l}.$$

Note that s_α and t_α are normalized quantities used to depict the span of a distribution's feasibility – the span of compatible $\boldsymbol{\theta}_\alpha$ vectors that a system can satisfy, within the context of our SPT setup - rather than direct measures of "shape" (e.g., skewness, or tail-width). Reference §6 for measures of shape. Note the following important observations:

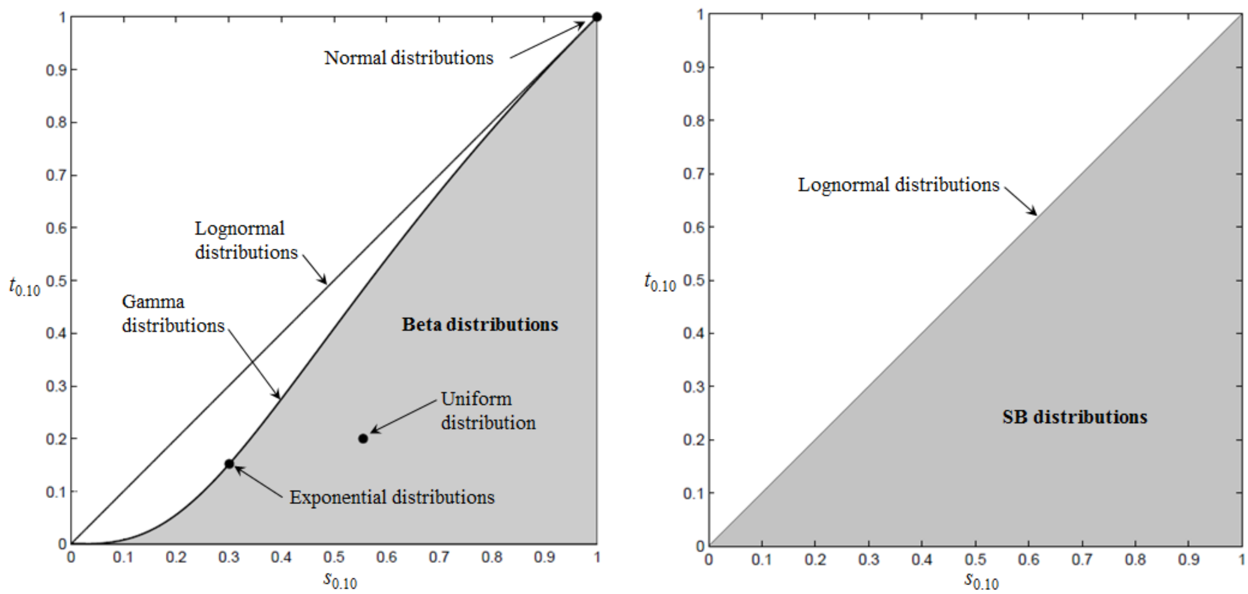
- s_α and t_α are invariant to simple changes of location and/or scale upon the distribution.

- Since l = Q(0), by the monotonicity of percentiles, both s_α and t_α are bounded between zero and one.
- For each $\alpha \in (0, 0.50)$, all non-degenerate univariate probability distributions live in the unit square defined by: $s_\alpha \in [0,1]$, $t_\alpha \in [0,1]$.

Figure 10 shows the span of several commonly-named distributions within the $\{s_\alpha, t_\alpha\}$ space for α = 0.10. Notice that the uniform and exponential distributions correspond to a single point within this space, since they lack shape parameters. Alternatively, the lognormal and gamma distribution systems each have a single shape parameter, and thus correspond to a curve within the $\{s_\alpha, t_\alpha\}$ space. In particular, the family of lognormal distributions lies along the line segment $s_\alpha = t_\alpha$ within the $\{s_\alpha, t_\alpha\}$ space, for any value of $\alpha \in (0, 0.50)$. Since the family of beta distributions (with unspecified upper bound) has two separate shape parameters, it is contained in the two-dimensional (shaded) region within the $\{s_\alpha, t_\alpha\}$ space, for each value of α. Finally, the SB distributions occupy the lower half of the unit square in Figure 10, defined by $s_\alpha > t_\alpha$. By contrast, the MF property of the J-QPD system implies that these distributions span the *entire* interior of the unit square defined by: $s_\alpha \in (0,1)$, $t_\alpha \in (0,1)$. We revisit the $\{s_\alpha, t_\alpha\}$ space in §5, where we compare the closeness (Criterion 4) of J-QPD distributions to a vast array of commonly-named distributions.

Figure 10. Span of Several Existing Distributions within the $\{s_\alpha, t_\alpha\}$ **Space for** α = 0.10

a) Some Commonly-Named Distributions **b) SB Distributions**



5. The Closeness of the J-QPD Distributions to Commonly-Named Distributions

Up to this point, we have shown that the J-QPD-B and J-QPD-S distributions satisfy Criteria 1 through 3, of those listed in §2. However, a natural question is how close the J-QPD distributions are to commonly-named distributions (Criterion 4). In this section, we compare (1) the J-QPD-B system to the beta distributions, which include gamma as a limiting case, assuming (without loss of generality) support on [0, 1], and (2) the J-QPD-S system to the beta-prime distributions, which include both gamma and inverse-gamma distributions as limiting cases, assuming (without loss of generality) support on $[0, \infty)$[12]. In both cases, we perform the comparison with respect to two common SPTs – the {10th, 50th, 90th} and the {5th, 50th, 95th} percentiles.

To give context, suppose that an expert provides {10th, 50th, 90th} or {5th, 50th, 95th} percentile assessments consistent with each commonly-named distribution, presumed to represent the "true" distribution. We then compare each commonly-named distribution to the corresponding J-QPD-B (J-QPD-S) assignment, sharing the same SPT, and support on [0, 1] ($[0, \infty)$).

[12] We do not compare the J-QPD-S distributions to the lognormal distributions, since we showed in §4 that the J-QPD-S system subsumes the entire lognormal family as a special case.

5.1 Closeness Metrics

In comparing each commonly-named distribution (assumed to be the "true" distribution) to the corresponding J-QPD assignment, we use three measures of closeness:

- APDM – The absolute percent difference in means by interdecile range: (P90-P10).
- APDV – The absolute percent difference in the variances with respect to the true variance.
- KS – The Kolmogorov-Smirnov (KS) distance.

When comparing the mean values for two different distributions, we deem it appropriate for the error measure to be invariant to changes of location and scale. Let μ denote the mean of the true (commonly-named) distribution, and μ' denote the mean of the corresponding J-QPD assignment. The APDM error measure is given by:

$$APDM = 100 \cdot \left| \frac{\mu' - \mu}{x_{90} - x_{10}} \right|. \quad (11)$$

Notice that the APDM error measure, as defined, is invariant to location and scale, as desired. It is more common to divide by the standard deviation, σ, of the true distribution. However, we use $x_{90} - x_{10}$ as the normalizing measure of spread since σ is not typically known in practice, and since the standard deviations of the true and assigned (J-QPD) distributions will be different – recall that we are matching percentiles, and not standard deviations.

Alternatively, when comparing the variances of two distributions, we can remove location and scale by simply comparing with respect to the variance of the true distribution. Thus, using v and v' to denote the two variances in analogous fashion, the APDV error measure is given by:

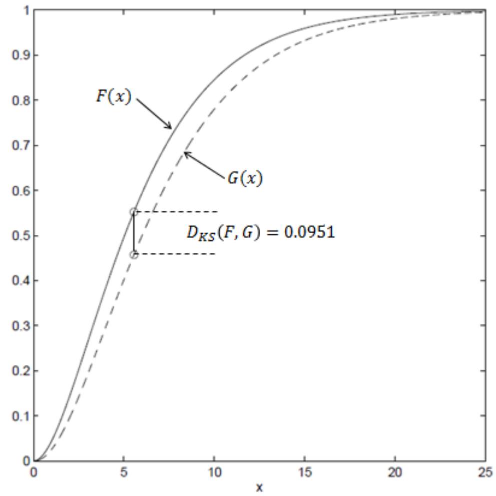
$$APDV = 100 \cdot \left| \frac{v' - v}{v} \right|. \quad (12)$$

Finally, we now briefly describe the KS distance between two distributions. For two separate cdfs, denoted $F(x)$ and $G(x)$, the KS distance between $F(x)$ and $G(x)$ is given by:

$$D_{KS}(F,G) = \sup_x | F(x) - G(x) |. \quad (13)$$

The KS distance is the largest absolute vertical deviation between $F(x)$ and $G(x)$, as depicted in Figure 11.

Figure 11. Illustration of the KS Distance between cdfs *F* and *G*.

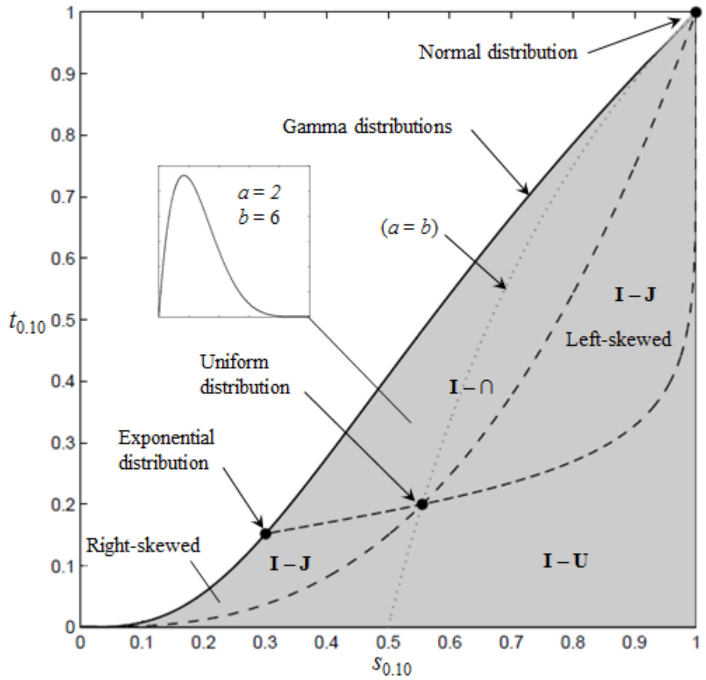


5.2 Methodology for Comparing J-QPD to Commonly-Named Distributions

Figure 12 displays the span of the beta distributions within the $\{s_{0.10}, t_{0.10}\}$ space. Note that each point within the shaded region corresponds to a specific beta distribution; i.e., a specific pair of {*a*, *b*} parameters. Observe that the beta distributions are partitioned into several key subfamilies – ∩-shaped (region I-∩, where $a \geq 1$, $b \geq 1$), U-shaped (region I-U, where $a < 1$, $b < 1$), right-skewed J-shaped (region I-J, with $a < 1$, $b \geq 1$), and left-skewed J-shaped (region I-J, with $a \geq 1$, $b < 1$). The uniform distribution ($a = 1$, $b = 1$) occurs as the intersection of the four subfamilies, while the exponential distribution is a special case of the gamma distributions, and occurs as a limiting case within the right-skewed J-shaped distributions. Finally, observe the dotted curve of symmetric distributions ($a = b$).

We construct a grid of approximately 104,000 points covering the feasible region for the beta distributions shown in Figure 12, spaced 0.002 in each dimension. For each such point, we identify the corresponding beta distribution by solving for the corresponding {*a*, *b*} parameter pair. Next, we compute $\boldsymbol{\theta}_\alpha = (0, x_\alpha, x_{0.50}, x_{1-\alpha}, 1)$ for this beta distribution, and then construct the QF for the corresponding J-QPD-B distribution, parameterized by $\boldsymbol{\theta}_\alpha$. Then, we compute the mean and variance of the beta distribution, as well as the mean and variance of the corresponding J-QPD-B assignment, and subsequently compute the APDM and APDV errors using the expressions given in §5.1. Finally, we evaluate the KS distance between the beta distribution, and its corresponding J-QPD-B assignment.

Figure 12. Feasible Region of the Beta Distributions in the $\{s_{0.10}, t_{0.10}\}$ Space



5.3 The Closeness of J-QPD-B Distributions to Commonly-Named Distributions

Before providing our comprehensive comparison of the J-QPD-B distributions with respect to the family of beta distributions, we provide several examples in order to lend context and guide intuition. Figure 13 and Figure 14 provide pdfs and cdfs, respectively, for nine J-QPD-B distributions, each parameterized by $\boldsymbol{\theta}_{0.10} = (l, x_{0.10}, x_{0.50}, x_{0.90}, u)$ for some commonly-named distributions. In addition, Figure 14 provides the APDM, APDV, and KS error measures in each case.

With the exception of the two triangular distributions, notice that the pdf of each J-QPD-B distribution (dashed) is barely discernible from the named distribution (solid). The cdfs are nearly indiscernible in all nine cases. Except for the two highly-skewed J-shaped beta distributions (beta(10, 1) and beta(0.7, 5)), all APDM (APDV) errors are less than 0.03% (1%). Except for the two triangular distributions, KS distances are no greater than 0.0035.

Figure 13. J-QPD-B (dashed) Parameterized by $\theta_{0.10}$ for Some Named Distributions (solid) – PDFs

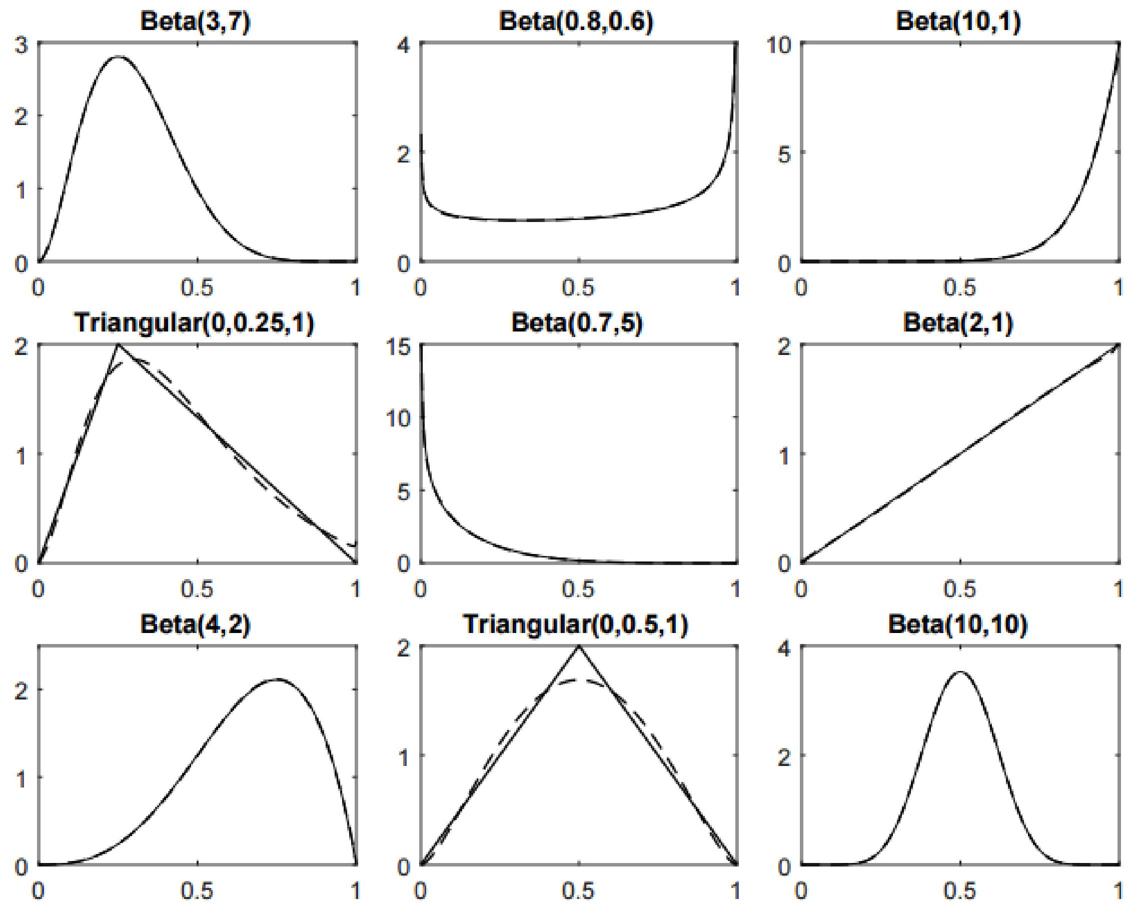
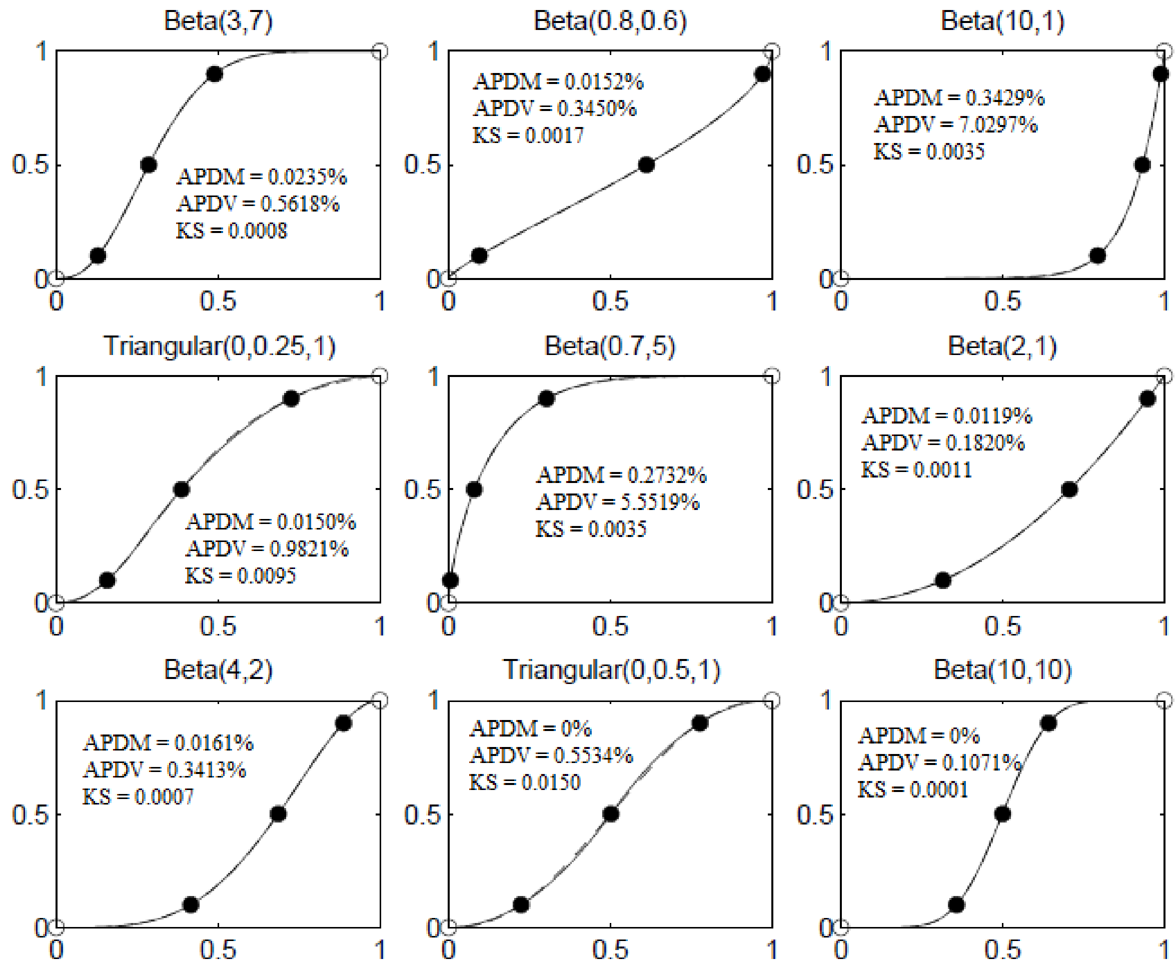


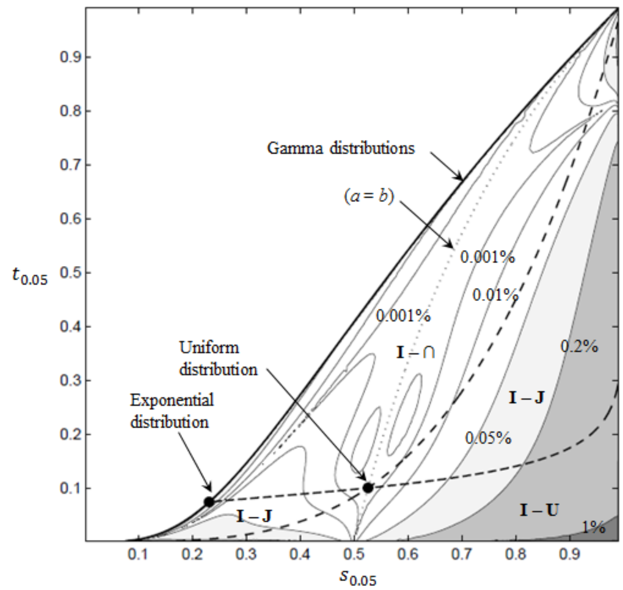
Figure 14. J-QPD-B (dashed) Parameterized by $\theta_{0.10}$ for Some Named Distributions (solid) – CDFs



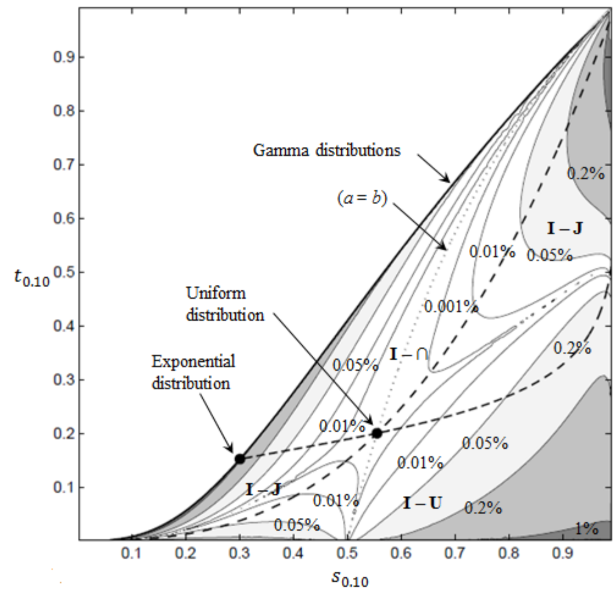
We now compare the J-QPD-B distributions to our equally spaced grid of roughly 104,000 beta distributions, covering the region depicted in Figure 12. Figure 15 depicts the span of the beta distributions within the $\{s_{0.05}, t_{0.05}\}$ ($\{s_{0.10}, t_{0.10}\}$) space, along with shaded error contours of APDM, APDV, and KS distances of the J-QPD-B distributions with respect to the corresponding beta distribution sharing the same SPT. Note that due to the presence of two separate shape parameters, the beta distributions occupy a region for each α within the $\{s_\alpha, t_\alpha\}$ space, while the gamma distributions occupy a curve at the boundary.

Figure 15. Error Measures of J-QPD-B w.r.t. Beta Distributions: $\theta_{0.05}$ (left) and $\theta_{0.10}$ (right)

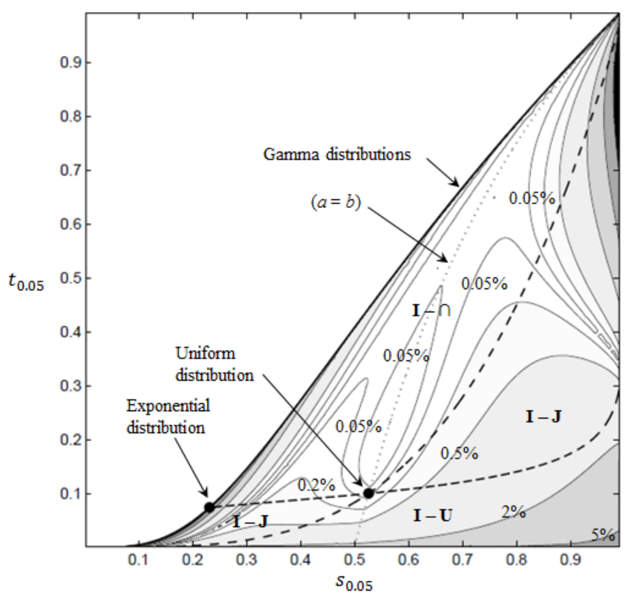
a) APDM using $\boldsymbol{\theta}_{0.05} = (0, x_{0.05}, x_{0.50}, x_{0.95}, 1)$



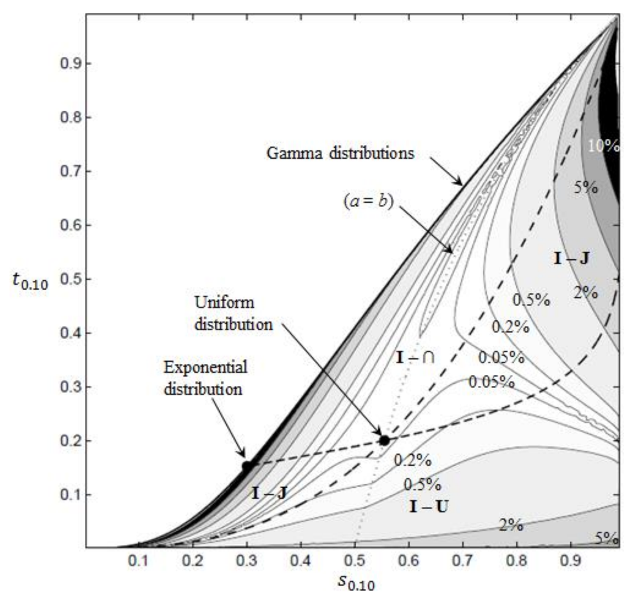
b) APDM using $\boldsymbol{\theta}_{0.10} = (0, x_{0.10}, x_{0.50}, x_{0.90}, 1)$



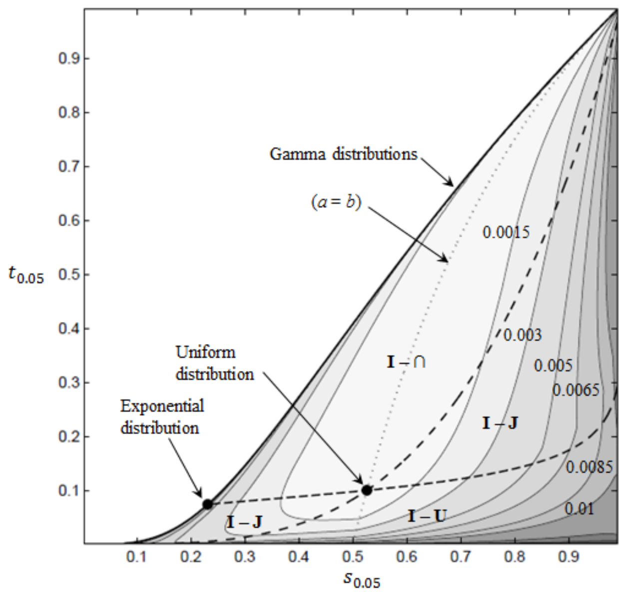
c) APDV using $\boldsymbol{\theta}_{0.05} = (0, x_{0.05}, x_{0.50}, x_{0.95}, 1)$



d) APDV using $\boldsymbol{\theta}_{0.10} = (0, x_{0.10}, x_{0.50}, x_{0.90}, 1)$



e) KS distance using $\boldsymbol{\theta}_{0.05} = (0, x_{0.05}, x_{0.50}, x_{0.95}, 1)$



f) KS distance using $\boldsymbol{\theta}_{0.10} = (0, x_{0.10}, x_{0.50}, x_{0.90}, 1)$

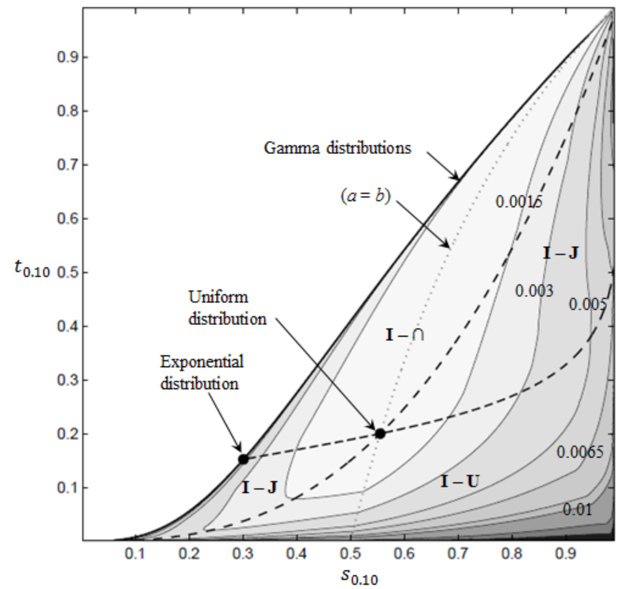


Table 1 provides summary statistics for each error measure across our grid of 104,000 points in Figure 15. For all three error measures, values are the worst (greatest) for bimodal and/or highly skewed J-shaped distributions. For the ∩-shaped beta distributions, for instance, errors grow largest as we approach the exponential distribution at the boundary in Figure 15. For the I-∩ region as-a-whole, however, notice that APDM (APDV) values are generally less than 0.2% (5%), values which are comparable to the "Beta (3,7)" panel of Figure 14. KS distances for the I-∩ region are generally less than 0.003, a value comparable to the "Beta (10,1)" panel of Figure 14.

Table 1. Error Measures for J-QPD-B Distributions w.r.t. Beta Distributions (by region)

	Minimum	Median	Maximum	Mean
I-∩ (Beta)				
APDM $(\boldsymbol{\theta}_{0.05})$	0.0%	0.0%	0.6%	0.0%
APDM $(\boldsymbol{\theta}_{0.10})$	0.0%	0.0%	1.6%	0.1%
APDV $(\boldsymbol{\theta}_{0.05})$	0.0%	0.1%	51.7%	0.3%
APDV $(\boldsymbol{\theta}_{0.10})$	0.0%	0.3%	46.6%	1.1%
KS distance $(\boldsymbol{\theta}_{0.05})$	0.0	0.0	0.0	0.0
KS distance $(\boldsymbol{\theta}_{0.10})$	0.0	0.0	0.0	0.0
I – J(Beta)				
APDM $(\boldsymbol{\theta}_{0.05})$	0.0%	0.1%	2.5%	0.2%
APDM $(\boldsymbol{\theta}_{0.10})$	0.0%	0.0%	9.7%	0.2%
APDV $(\boldsymbol{\theta}_{0.05})$	0.0%	0.6%	445.0%	2.3%
APDV $(\boldsymbol{\theta}_{0.10})$	0.0%	1.0%	809.0%	4.1%

KS distance $(\boldsymbol{\theta}_{0.05})$	0.0	0.0	0.0	0.0
KS distance $(\boldsymbol{\theta}_{0.10})$	0.0	0.0	0.0	0.0
$\mathrm{I-U(Beta)}$				
APDM $(\boldsymbol{\theta}_{0.05})$	0.0%	0.3%	2.0%	0.4%
APDM $(\boldsymbol{\theta}_{0.10})$	0.0%	0.1%	2.2%	0.2%
APDV $(\boldsymbol{\theta}_{0.05})$	0.0%	1.7%	9.8%	2.0%
APDV $(\boldsymbol{\theta}_{0.10})$	0.0%	0.7%	9.3%	1.1%
KS distance $(\boldsymbol{\theta}_{0.05})$	0.0	0.0	0.1	0.0
KS distance $(\boldsymbol{\theta}_{0.10})$	0.0	0.0	0.1	0.0

Errors are generally larger for the I-J and I-U regions, as compared to the I-∩ region. Notice that for the APDM and APDV error measures in Table 1, error values increase rapidly for distributions at the boundaries of both the I-J and I-U regions (see maximum values), while overall errors across these regions are small (based on median values). For example, for the I-J region, we find that errors only increase rapidly as $a \to 0$ and $b \to \infty$, or vice versa. Such distributions exist at the lower left and upper right corners of the region depicted in Figure 15.

5.4 The Closeness of J-QPD-S Distributions to Commonly-Named Distributions

This section parallels §5.3 in structure, except that we now compare the J-QPD-S distributions to the beta-prime distributions. We first build context with several examples. Figure 16 (Figure 17) provides pdfs (cdfs) for six J-QPD-S distributions, each parameterized by $\boldsymbol{\theta}_{0.10} = (l, x_{0.10}, x_{0.50}, x_{0.90})$ for the pdfs (cdfs) of the six respective commonly-named distributions shown, all having semi-infinite support. In addition, Figure 17 provides the APDM, APDV, and KS error measures in each case.

As with the J-QPD-B comparisons, each J-QPD-S distribution (dashed) is barely discernible from the named distribution (solid), particularly with respect to cdfs. Notice that all three error measures are zero for the lognormal case shown in Figure 16 and Figure 17, since we established that the J-QPD-S distributions subsume the lognormal distributions as a special case. The largest errors occur in the case of the more skewed, J-shaped distributions, as shown in the "Exponential (1)", "Weibull (10,0.5)", and "Gamma (0.5,1)" cases.

Figure 16. J-QPD-S (dashed) Parameterized by $\theta_{0.10}$ for Some Named Distributions (solid) – PDFs

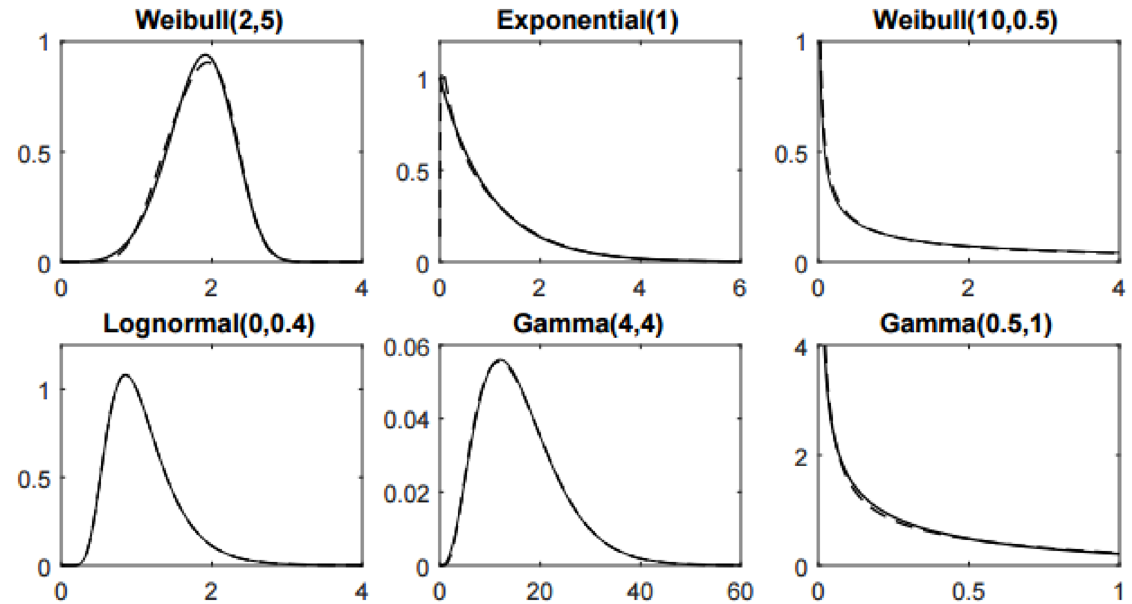
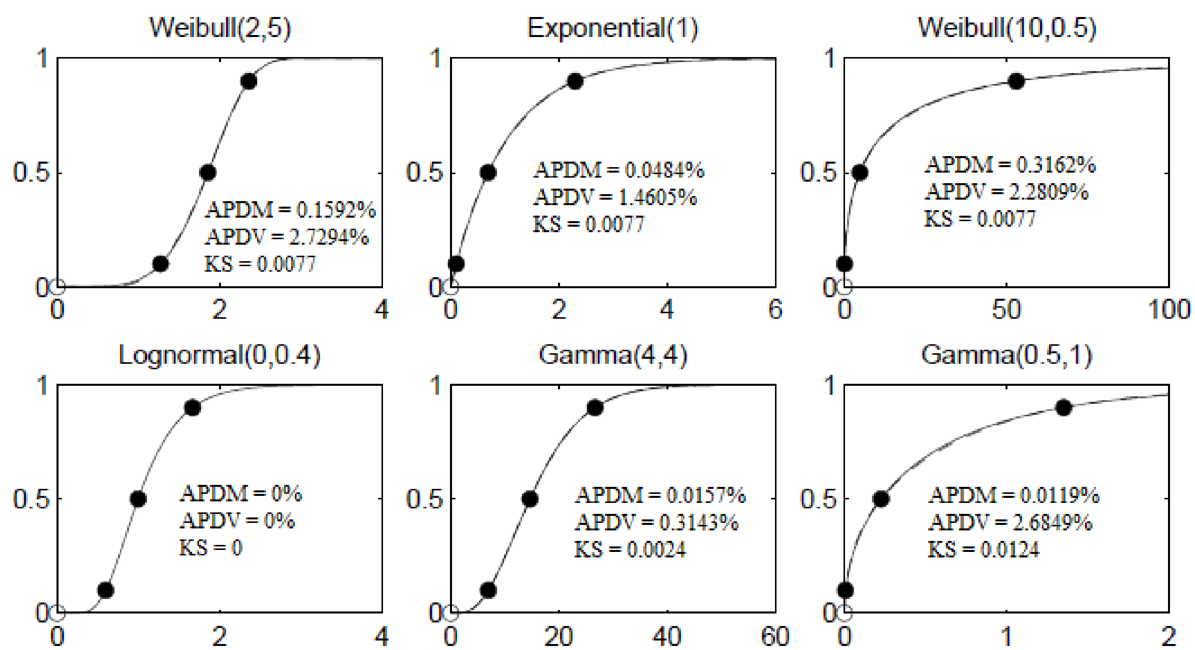


Figure 17. J-QPD-S (dashed) Parameterized by $\theta_{0.10}$ for Some Named Distributions (solid) – cdfs

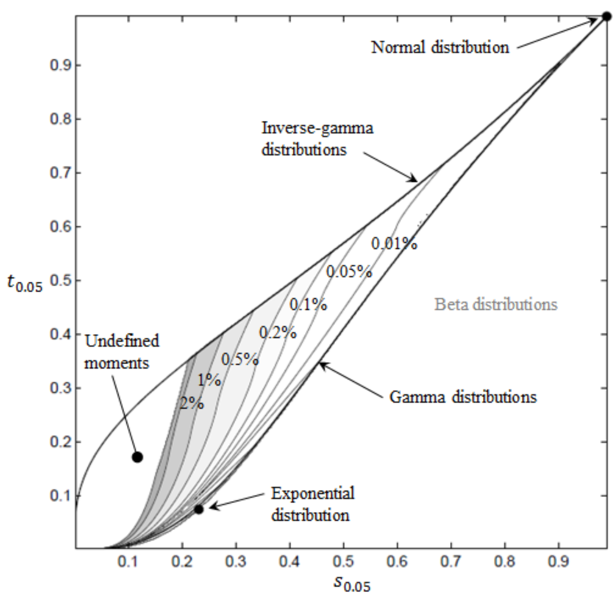


We now compare the J-QPD-S distributions to the beta-prime distributions, following an analogous procedure to that discussed in §5.2 for beta-prime distributions. In particular, we construct a grid of approximately 35,000 points across the feasible region for the beta-prime distributions, spaced 0.002 in each dimension. Figure 18 depicts the feasible region of the beta-prime distributions within the $\{s_{0.05}, t_{0.05}\}$

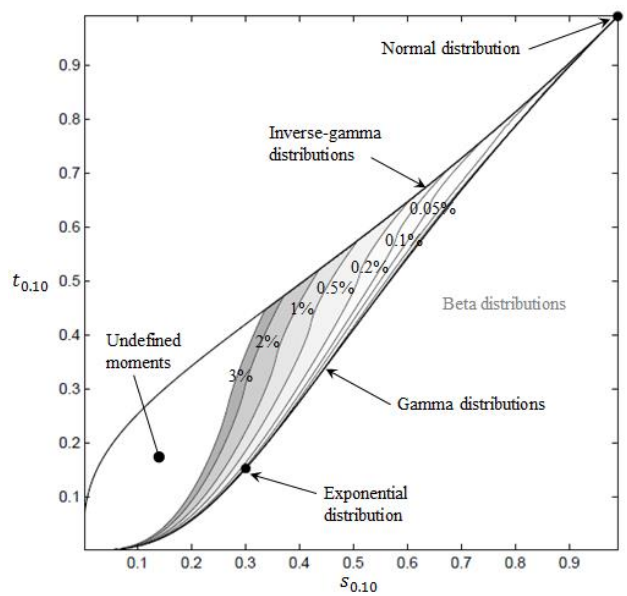
($\{s_{0.1}, t_{0.1}\}$) space, along with shaded error contours of APDM, APDV, and KS distances of the J-QPD-S distributions with respect to the corresponding beta-prime distribution sharing the same SPT. Similar to the beta distributions, the beta-prime distributions have two separate shape parameters, and thus occupy a region for each α within the $\{s_\alpha, t_\alpha\}$ space, whereas the gamma and inverse-gamma distributions each occupy a curve at the boundary.

Figure 18. Error Measures of J-QPD-S w.r.t. Beta-Prime Distributions: $\boldsymbol{\theta}_{0.05}$ (left) and $\boldsymbol{\theta}_{0.10}$ (right)

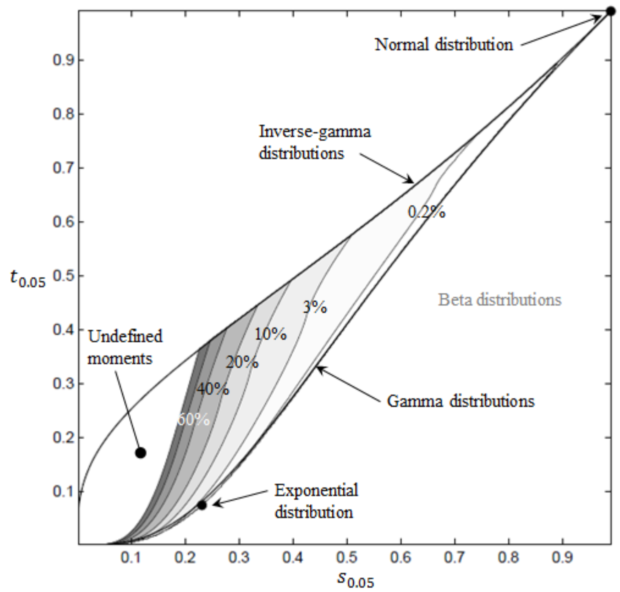
a) APDM using $\boldsymbol{\theta}_{0.05} = (0, x_{0.05}, x_{0.50}, x_{0.95}, 1)$



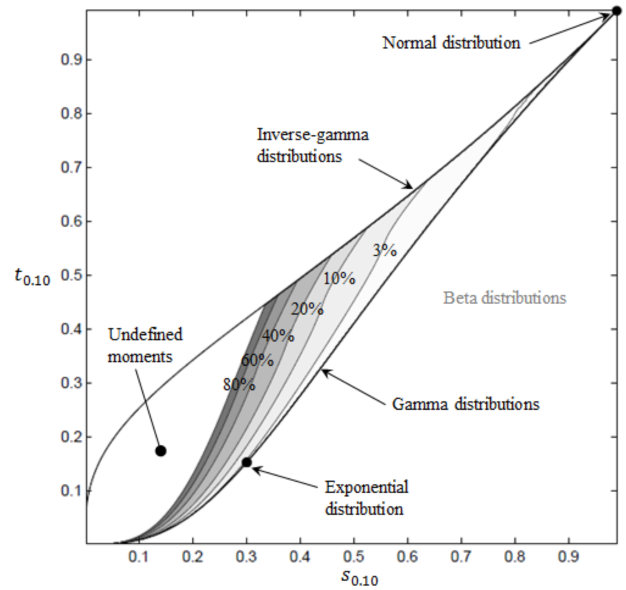
b) APDM using $\boldsymbol{\theta}_{0.10} = (0, x_{0.10}, x_{0.50}, x_{0.90}, 1)$



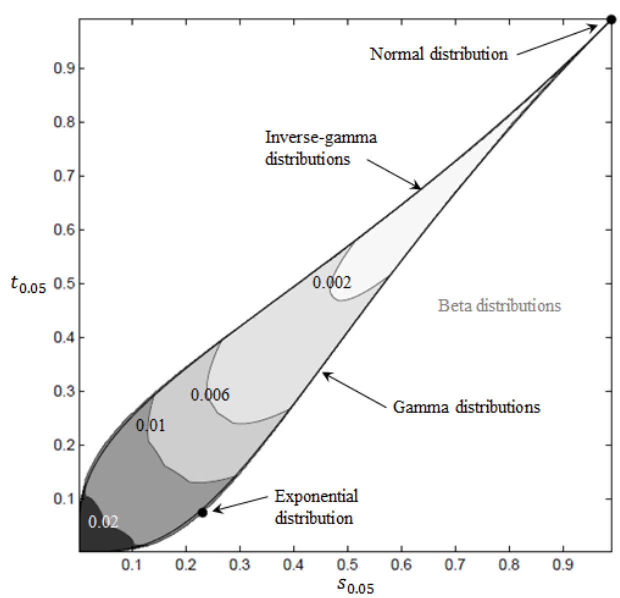
c) APDV using $\boldsymbol{\theta}_{0.05} = (0, x_{0.05}, x_{0.50}, x_{0.95}, 1)$



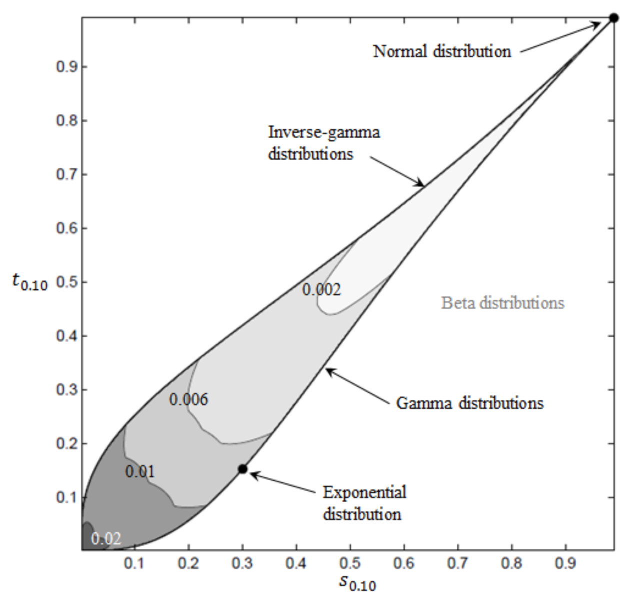
d) APDV using $\boldsymbol{\theta}_{0.10} = (0, x_{0.10}, x_{0.50}, x_{0.90}, 1)$



e) KS distance using $\boldsymbol{\theta}_{0.05} = (0, x_{0.05}, x_{0.50}, x_{0.95}, 1)$



f) KS distance using $\boldsymbol{\theta}_{0.10} = (0, x_{0.10}, x_{0.50}, x_{0.90}, 1)$



Notice that in Figure 18 panels a through d, there exists a white region within the span of the beta-prime system shown, labeled “undefined moments”. In particular, the mean and variance of these beta-prime distributions are undefined, as are the higher-order moments. Thus, we exclude them from our comparison to J-QPD-S distributions with respect to mean (APDM) and variance (APDV). However, we include these distributions when comparing with respect to KS, since the KS distance between any two distributions is necessarily bounded between zero and one.

Table 2 provides summary statistics for each error measure across our grid of roughly 35,000 points in Figure 18. For all three error measures, values grow upwards as both s_α and t_α decrease toward zero. For the APDM and APDV measures, the errors increase rapidly as we approach the "undefined moments" region shown, thus yielding the large maximum values shown for these measures in Table 2. This is simply because while J-QPD-S distributions have finite moments, the moments of the beta-prime distribution diverge upwards near the "undefined moments" boundary. However, notice that for most of the region shown in panels a and b of Figure 18, APDM errors are generally less than 2%. For the ∩-shaped gamma distributions, shown along the "gamma distributions" curve between the "exponential distribution" and "normal distribution" points, we evaluate the worst-case APDM error to be only 0.196% for $\alpha = 0.05$, and only 0.048% for $\alpha = 0.1$ – both corresponding to the exponential distribution.

Table 2. Error Measures for J-QPD-S Distributions w.r.t. Beta-Prime Distributions

	Minimum	Median	Maximum	Mean
APDM $(\boldsymbol{\theta}_{0.05})$	0.0%	0.1%	6.9%	0.5%
APDM $(\boldsymbol{\theta}_{0.10})$	0.0%	0.5%	70.5%	1.2%
APDV $(\boldsymbol{\theta}_{0.05})$	0.0%	5.2%	180.4%	17.3%
APDV $(\boldsymbol{\theta}_{0.10})$	0.0%	11.2%	1.6e+03%	24.6%
KS distance $(\boldsymbol{\theta}_{0.05})$	0.0	0.0	0.0	0.0
KS distance $(\boldsymbol{\theta}_{0.10})$	0.0	0.0	0.0	0.0

For the APDV measure, we evaluate the worst-case error for the bell-shaped gamma distributions to be only 0.742% for $\alpha = 0.05$, and 1.461% for $\alpha = 0.1$. As with ADPM errors, APDV values grow large near the "undefined moments" boundary line. Finally, regarding the comparisons in panels "e" and "f" of Figure 18, notice that KS distances are generally less than 0.02 for the entire region. To lend context, the mean value of 0.0075 (0.006) shown in Table 2 and corresponding to panel e (f) is similar in magnitude to the "Weibull (2,5)", "Exponential(1)", and "Weibull (10,0.5)" examples provided in Figure 16 and Figure 17.

The main takeaway of this section is the notable conformity of the J-QPD-B and J-QPD-S distributions to Criteria 4, as measured by the APDM, APDV, and KS accuracy metrics. We now examine the flexibility of the J-QPD-B (-S) systems.

6. The Flexibility of the J-QPD Distributions

We now examine the J-QPD distributions with respect to Criterion 5 – flexibility. Specifically, we refer to the span of a system within the skewness-kurtosis space developed by Pearson (1895, 1901, 1916), due to its conventional use.[13] For alternative, percentile-based measures of flexibility, see (for example) Moors (1988) and Moors et al. (1996) in the statistics literature, and Powley (2013) in the decision analysis literature.

We first point out the distinction in the role of our $\{s, t\}$ space used in §4 and §5, compared to Pearson's $\{\beta_1, \beta_2\}$ (i.e., squared-skewness, kurtosis) space. The latter refers to conventional measures of shape, whereas s and t are normalized quantities used to depict the span of a distribution's feasibility – the span of compatible $\boldsymbol{\theta}_\alpha$ vectors that a system can satisfy, within the context of our SPT setup[14] - rather than direct measures of "shape" (e.g., skewness, or tail-width).

Figure 19 left (right) shows the span of the J-QPD-B (J-QPD-S) distributions in light grey within the $\{\beta_1, \beta_2\}$ space, over the region considered by Pearson (1895, 1901, 1916), Johnson (1949), as well as Hammond and Bickel (2013a and 2013b). Note that in both cases, we overlay Pearson's [Type] distributions, as well as the curve of lognormal distributions. We first note that J-QPD-B distributions span the entirety of the $\{\beta_1, \beta_2\}$ space. Also, since J-QPD-B distributions possess three shape parameters for each value of α, each point in the $\{\beta_1, \beta_2\}$ space may have more than one location-scale J-QPD-B distribution corresponding to the given point.

Alternatively, we see in Figure 19 (right) that the J-QPD-S distributions partially span the $\{\beta_1, \beta_2\}$ space, partly due to the arc-sinh operator, which results in finite moments. Also, J-QPD-S distributions have two shape parameters for each value of α, as opposed to three in the case of J-QPD-B. It is important

[13] In the statistics literature, see: Ord (1972); Johnson (1949); Johnson, Kotz, and Balakrishnan (1994); and Tadikamalla and Johnson (1982). In the decision analysis literature, see (for example), Hammond and Bickel (2013a and 2013b), and Keelin (2016).
[14] The authors are indebted to Tom Keelin, for thoughtful discussion on the distinction between feasibility and *flexibility*, as well as to the reviewers for their related comments.

to note that while J-QPD-S is maximally-*feasible*, meaning that it can accommodate any θ_α vector, the fatness of its tails (kurtosis) is approximately limited by lognormal distributions. However, this does not severely limit its usefulness in practice, since J-QPD-S covers all of the familiar bell-shaped beta distributions in the $\{\beta_1, \beta_2\}$ space, and covers most of the J-shaped region, including the exponential distribution, as well as subsumes the normal and lognormal distributions. Recall that J-QPD-S can also closely match the Gamma and Weibull distributions, and has the added ability to achieve bimodal forms.

Figure 19. Span of J-QPD-B (left) and J-QPD-S (right) Distributions within Pearson's $\{\beta_1, \beta_2\}$ Space

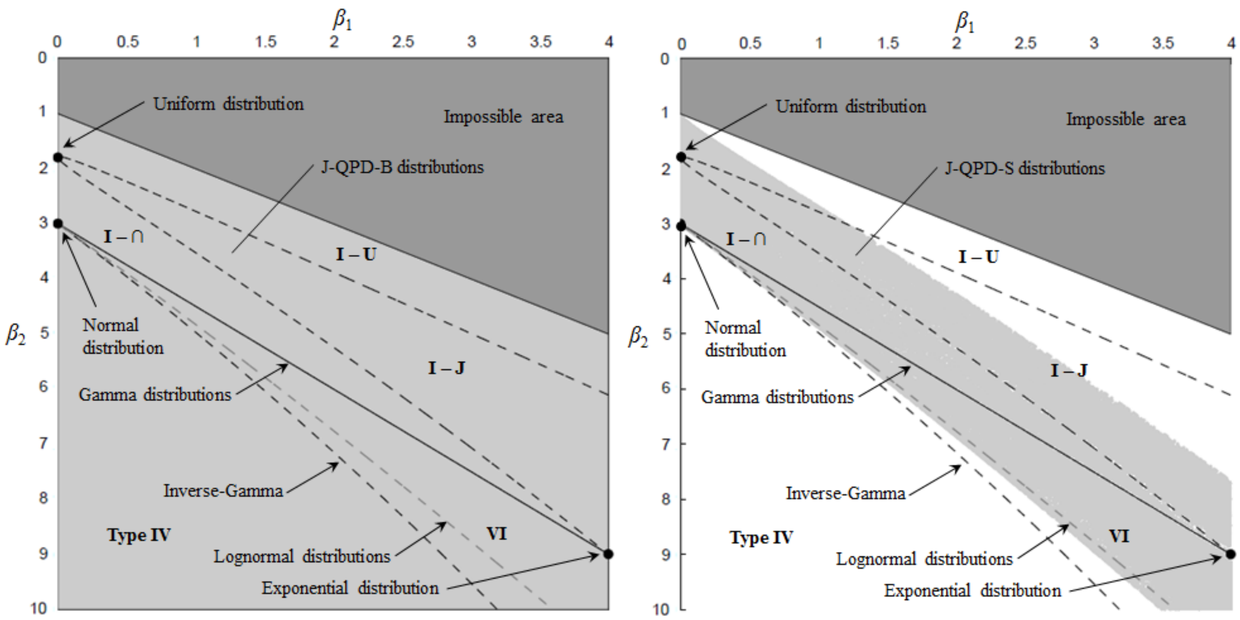


Table 3 (Table 4) lists several distribution subsystems of J-QPD-B (J-QPD-S) that occur as parametric limiting cases, characterized in standard (no location or scale parameters) quantile function form, along with associated comments. Perhaps the most notable of these is the S-II subfamily, since it is the only subfamily that can be expressed in our SPT-QPD form, and is also maximally-feasible. Given $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, \infty)$, where l is the finite lower bound of support, the resulting location-scale QF for the J-QPD-S II (or S-II) distributions, is as follows:

$$Q_{S\text{-II}}(p) = l + \theta \exp\left(\lambda \sinh\left(\delta\left(\Phi^{-1}(p) + nc\right)\right)\right), \tag{14}$$

where,

$$c = \Phi^{-1}(1-\alpha),$$
$$L = \log(x_\alpha - l),\ B = \log(x_{0.5} - l),\ H = \log(x_{1-\alpha} - l),$$
$$n = \mathrm{sgn}(L + H - 2B)$$

$$\theta = \begin{cases} x_{\alpha} - l, \ n = 1 \\ x_{0.50} - l, \ n = 0 \\ x_{1-\alpha} - l, \ n = -1 \end{cases}$$

$$\delta = \left(\frac{1}{c}\right)\left(\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right)\right),$$

$$\lambda = \left(\frac{1}{\sinh(\delta c)}\right)\min(H-B, B-L).$$

Note that only S-II have unbounded moments, and correspond to the special case of a J-QPD-B distribution in which $(l, \mathbf{x}_{\alpha})$ are fixed finite values, and where the upper bound (u) approaches infinity. S-II distributions satisfy desiderata 1 through 4, but are not applicable to the $\{\beta_1, \beta_2\}$ space, due to their unbounded moments. The S-II distributions have semi-bounded support, and two shape parameters, and essentially correspond to the J-QPD-S distributions with the arc-sinh operator removed. The S-II distributions might be well-suited for modelling uncertainties with heavy, Pareto-type tails. Figure 20 (Figure 21) provides CDFs (PDFs) for several S-II distributions (dashed), as compared to the corresponding J-QPD-S distribution (solid) sharing the same $\boldsymbol{\theta}_{\alpha}$ vector in each case. Notice that the S-II distributions are generally a "more heavy-tailed version" of J-QPD-S distributions, but can sometimes produce different shapes than J-QPD-S, as shown in Figure 21 for the case in which $\boldsymbol{\theta}_{0.1} = (0, 10, 20, 25, \infty)$.

Table 3. Limiting Distributions within the J-QPD-B System

Name	Quantile function	Parameters	Also contains	Comments
S-II	$\exp\left(\lambda \cdot \sinh\left(\delta\left(\pm c + \Phi^{-1}(p)\right)\right)\right)$	$\lambda > 0,\ \delta > 0$	Lognormal, normal distributions	Moments are unbounded except in lognormal/normal cases
U-I	$\sinh\left(\delta\left(\pm c + \Phi^{-1}(p)\right)\right)$	$\delta > 0$	Normal distributions	Special case of Johnson distributions
B-II	$\Phi\left(\xi + \lambda \cdot \Phi^{-1}(p)\right)$	$\lambda > 0,\ \xi \in \mathbb{R}$	Lognormal, normal, uniform distributions	"Probit-Normal" distributions

Table 4. Limiting Distributions within the J-QPD-S System

Name	Quantile function	Parameters	Also contains	Comments

H-II	$\exp\left(\lambda\cdot\left(\Phi^{-1}(p)\pm\left\|\Phi^{-1}(p)\right\|\right)\right)$	$\lambda>0$		
S-III	$\exp\left(\sigma\cdot\Phi^{-1}(p)\right)$	$\sigma>0$	Normal distributions	Lognormal distributions
U-II	$\sinh\left(\sinh^{-1}\left(\delta\,\Phi^{-1}(p)\right)\pm\sinh^{-1}(c\delta)\right)$	$\delta>0$	Normal distributions	

Figure 20. Comparison of Several S-II and J-QPD-S CDFs with the Same $\boldsymbol{\theta}_\alpha$

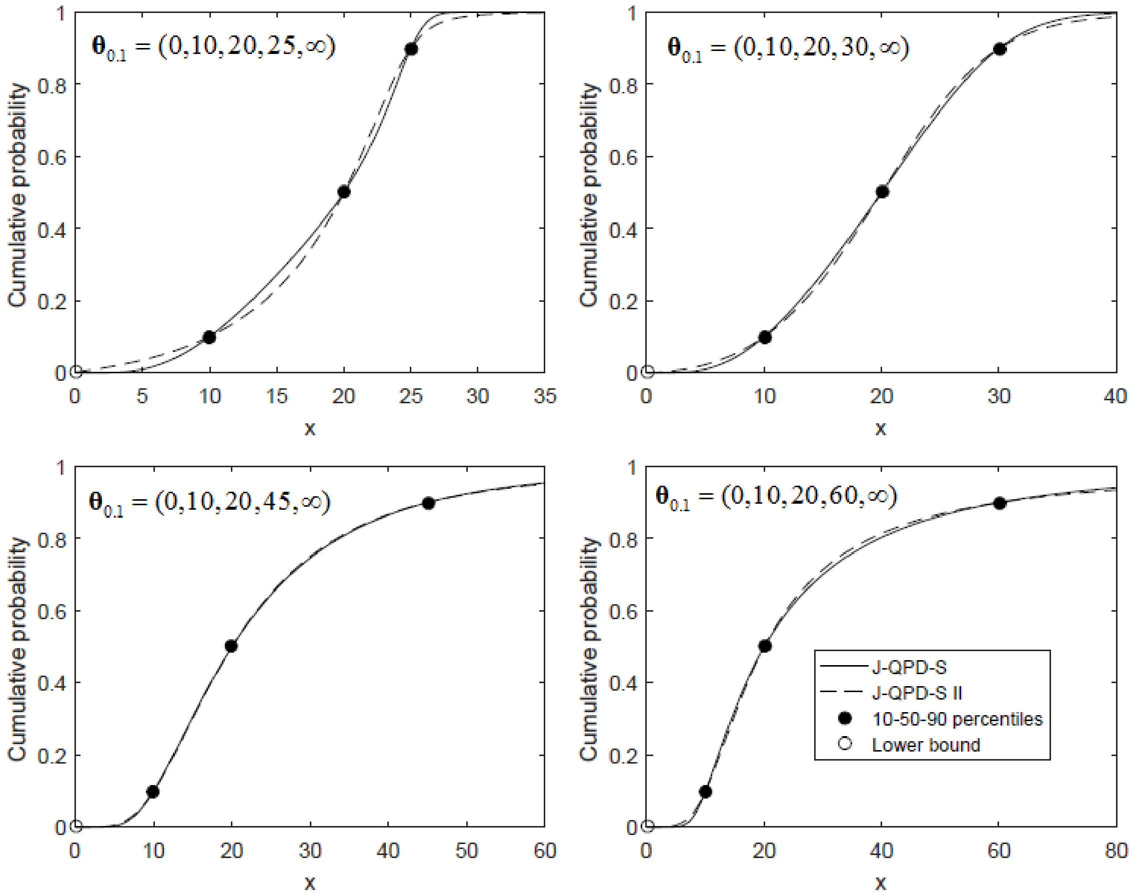
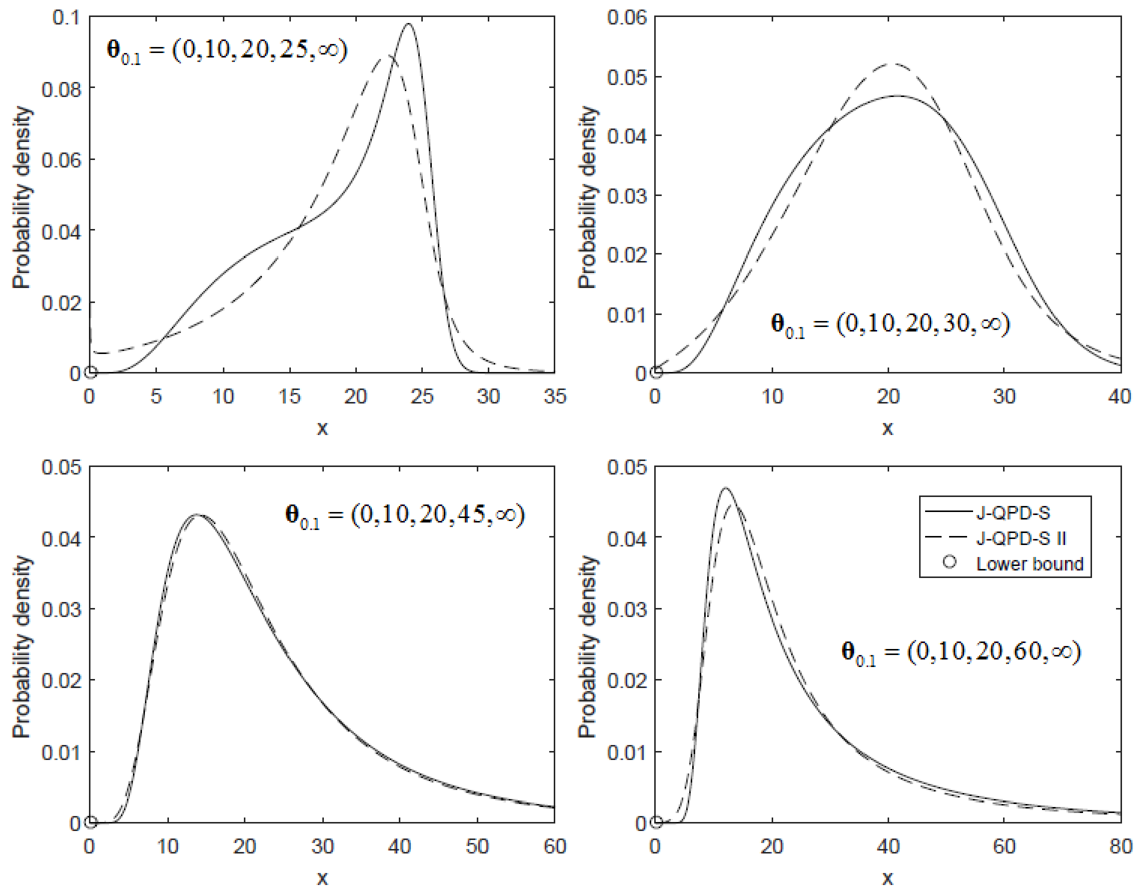


Figure 21. Comparison of Several S-II and J-QPD-S PDFs with the same $\boldsymbol{\theta}_\alpha$



The remaining subfamilies in Table 3 and Table 4 are not amenable to our QPD setup, but are worth mentioning briefly. The U-I distributions have unbounded support, a single shape parameter, and correspond to a special case of the Johnson-SU distributions. The, B-II distributions correspond to the “probit-normal” distributions, which have bounded support and two shape parameters, and are similar to the logit-normal distributions proposed by Mead (1965). In Table 4, the S-III distributions simply correspond to the lognormal distributions. H-II represent hybrid distributions having a point mass at one end of support, and continuity elsewhere. S-II distributions are bounded (semi-bounded) when “+/-“ is replaced with “-“ (“+”). Finally, U-II are bell-shaped distributions with one shape parameter (affecting both skewness and kurtosis) and unbounded support.

7. A Decision Analysis Evaluation Using the J-QPD Distribution System

To fully demonstrate the usefulness of the J-QPD system, we apply it to a slightly revised version of an example used by Powley (2013). Our goal is to lend further credence to the J-QPD system as a tool for accurately encoding an expert's knowledge, and to demonstrate the convenience of the J-QPD system in the decision modelling process.

7.1 The Decision Problem

A pharmaceutical company wants to evaluate two possible alternatives for a new drug candidate, which has just received FDA approval:

(1) *Market* the new drug.

(2) *License* the marketing rights of the new drug to a competing company, in exchange for an upfront payment, along with continuing royalties.

The value measures represent the net present value (NPV) of profit discounted over a ten-year time horizon for the respective alternatives. The decision diagram, deterministic inputs, and assessments, are given in Figure 22, Table 5, and Table 6. Finally, the company is risk-neutral, so that alternatives are evaluated based upon their expected NPV. For more detail of the original problem, see Powley (2013). We now evaluate the two alternatives.

Figure 22. The Company's Influence Diagram

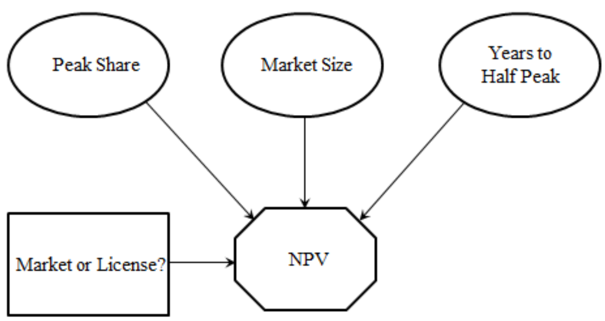


Table 5. The Company's Deterministic Inputs

time periods, T	unit selling price, p	unit cost, c	upfront payment, A	royalty rate, f
10 years	\$500	\$15	\$75 million	12%

Table 6. The Company's Uncertainty Assessments of Low-Base-High and Bound Values

	Units	10^{th}	50^{th}	90^{th}	Lower Bound	Upper Bound
Market Size (m)	# Rx	20,000	50,000	100,000	0	very large
Market Share (s_p)	%	60%	75%	90%	0%	100%
Years to Half-Peak (t_h)	years	2	3	5	1	very large

NOTE: Rx = number of prescriptions.

7.2 Evaluating the Market and License Alternatives

We now define the J-QPDs that we use to model this decision and then use simulation to compute the expected NPV.

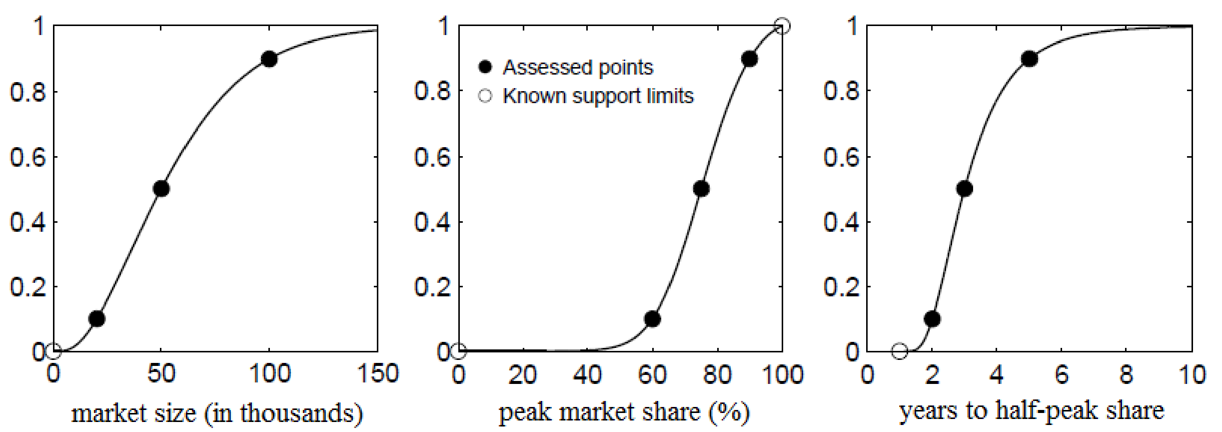
Quantifying Input Uncertainties

To evaluate the *Market* and *License* alternatives, we must first assign probability distributions to each of the three uncertainties. Given the bound remarks in Table 6, we apply a J-QPD-B distribution to *Market Share*, and apply J-QPD-S distributions to *Market Size* and *Years to Half-Peak*, where we treat "very large" as being unbounded above. Table 7 provides our precise J-QPD assignments for the three uncertainties, honoring the assessment data and bound remarks in Table 6. Notice that for *Years to Half-Peak*, we have the special case of a lognormal distribution. Figure 23 provides plots of the cdf for the J-QPD assignments in Table 7.

Table 7. The Analyst's J-QPD Assignments

Uncertainty	Type	QF	Bounds
Market Size (m)	J-QPD-S	$10^5 \cdot \exp\left(1.1753 \cdot \sinh\left(-0.5600 + \sinh^{-1}\left(0.4602 \cdot \Phi^{-1}(p)\right)\right)\right)$	$[0, \infty)$
Market Share (s_p)	J-QPD-B	$\Phi(0.2533 + 0.6015 \cdot \sinh(0.5094 \cdot (\Phi^{-1}(p) + 1.2816)))$	[0,1]
Years to Half-Peak (t_h)	J-QPD-S	$1 + 2 \cdot \exp\left(0.5409 \cdot \Phi^{-1}(p)\right)$	$[1, \infty)$

Figure 23. CDFs for the Input J-QPD Distribution Assignments



Simulation Procedure

In order to begin the evaluation process, we perform a simulation upon the two alternatives as follows:

(1) Use Latin-hypercube sampling to construct three large samples (of equal size) of i.i.d. uniform random variates, one vector for each input uncertainty.

(2) Perform inverse transform sampling by passing each vector of uniform random variates through each of the three QFs, so as to generate the observation vectors for: *Market Size*, *Peak Market Share*, and the *Years to Half-Peak*.

(3) Use the samples from (2) in conjunction with the value measures to generate observation vectors of NPV for both *Market* and *License* alternatives.

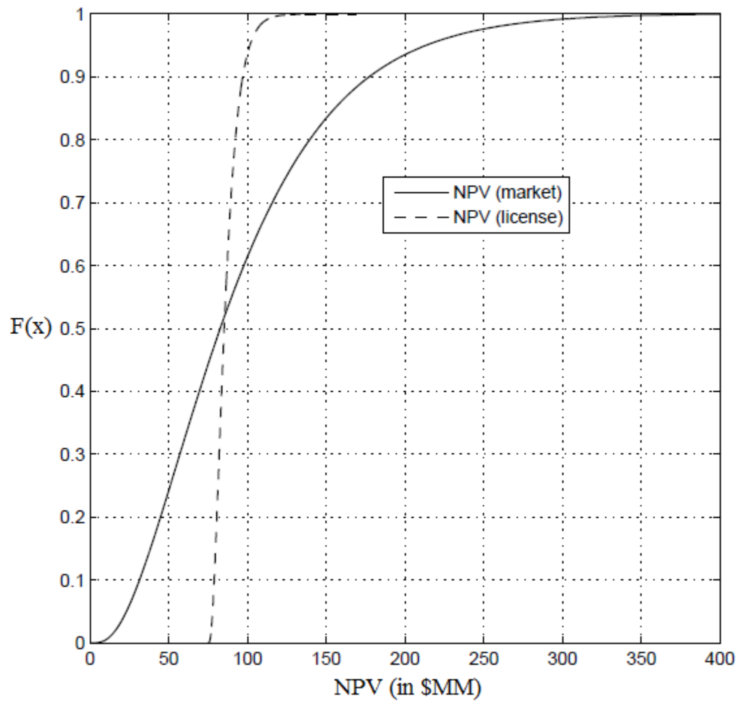
(4) Compute the mean NPV for both the *Market* and *License* alternatives.

Table 8 provides some key statistics generated from the simulation of the *Market* and *License* alternatives. The *Market* alternative has the largest expected NPV. Figure 24 provides the cdfs for the NPV of both alternatives.

Table 8. Simulation Output ($MM)

	Market	*License*
Mean (NPV)	96.1	86.9
x_{10}	31.5	78.9
x_{50}	82.8	85.2
x_{90}	177.9	97.0

Figure 24. CDFs for NPV for the Market and License Alternatives



Notice the overall simplicity of our evaluation procedure for the alternatives using the J-QPD system. We honor the assessment triplets for each uncertainty, and the corresponding assignments simply require computation of the distribution parameters in each case – no optimization (as with least-squares fitting) is needed. Finally, since our J-QPD distributions are inherently smooth, we obtain smooth output distributions, unlike those generated from discretization methods.

8. Conclusion and Recommendations for Practice

In this paper, we have developed the new J-QPD distribution system, by applying well-known transformations to the Johnson-SU distribution, followed by strategic re-parameterization. The resulting J-QPD system consists of two sub-families: the J-QPD-B (bounded) distributions, and the J-QPD-S (semi-bounded) distributions. Both systems are smooth, an attribute which we deem reasonable for capturing an expert's knowledge, as well as many real-world phenomena described by data. Unlike existing probability distribution systems, however, the J-QPD system satisfies all five of our desiderata presented in §2. In particular, the J-QPD system is conveniently parameterized by a symmetric percentile triplet (SPT) of low-base-high assessments (e.g., 10^{th}, 50^{th}, 90^{th} percentiles), as well as known support bounds, and is maximally-feasible, meaning that it can honor any SPT vector of coherent low-base-high values, along with any compatible pair (lower and upper) of support bounds. Both the QF and cdf are readily available. The J-QPD distributions generally have finite moments (similar to Johnson distributions), an attribute which we deem desirable for decision analysis practice, but also offer a highly flexible distribution with unbounded moments as a limiting case. J-QPD distributions are also highly flexible in the sense of Pearson. Finally, the J-QPD distributions can closely approximate distributions from a wide array of commonly-named families, when parameterized by the same SPT and support bounds.

In particular, J-QPD-S distributions subsume the entire lognormal family as a special case, and effectively serve as a two-shape-parameter extension to the lognormal family, but in quantile-parameterized form. In addition, J-QPD-S distributions are highly accurate at approximating Weibull and gamma distributions (including the exponential distribution), as well as most beta-prime distributions. The J-QPD-B distributions approximate nearly all beta distributions with high accuracy, particular the bell-shaped beta distributions, and also approximate triangular distributions with reasonable accuracy.

The J-QPD system provides a fast and convenient means of precisely assigning a continuous probability distribution to a triplet of low-base-high assessments, in conjunction with two possible support scenarios, commonly encountered in practice: known bounded support; semi-bounded support with known lower bound. The quantile function representations of the J-QPD distributions allow for direct Monte Carlo simulation via direct inverse transform sampling, as a means of generating output value distributions. These attributes collectively make J-QPD an important contribution to the decision analyst's toolkit.

Since J-QPD-S distributions exhibit lognormal tails, in the sense of Parzen (1979), we caution against their application to fatter-tailed distributions, such as Pareto-type distributions. If a semi-bounded distribution with heavy tails is needed, and finite moments are not required, then we suggest using the S-II (limiting distribution) provided in Table 3. Both J-QPD-B and J-QPD-S distributions are generally highly accurate at approximating bell-shaped, and modestly-skewed J-shaped distributions, as seen in §5.

The appropriateness of implementing the J-QPD system should be judged in connection with the characteristics of the decision being modelled. In situations where there is inherent non-linearity in the value function, as with (e.g.) network investment applications (pipelines, communications, power transmission, etc.), an analyst may choose to implement the smooth J-QPD distribution assignments as a supplement to more conventional approaches – such as using discretization shortcuts. Indeed, Keelin (2016) provides a compelling real-world bidding problem where use of discretization yields poor decision making.

In other cases, a decision maker may desire a quality estimate of the probability of incurring a loss (e.g., the probability that NPV is less than zero) in a riskier alternative – beyond simply having estimates of expected values. In this case, an analyst may prefer to implement the smooth J-QPD distribution assignments in order to better capture the shape (e.g., percentiles) of the output distribution.

We also offer some observations regarding the relative usefulness our new J-QPD systems compared to the systems developed by Keelin and Powley (2011) and Keelin (2016), using their more general notion of a QPD. For example, Keelin and Powley's QPDs are more directly extendible to a larger set of assessed quantile-probability pairs, and having a more general structure than SPT, while our J-QPDs specify a prescribed number of quantile-probability pairs of a specific structure. With that said, given our prescribed quantile-probability structure, our J-QPD system is maximally-feasible, whereas Keelin and Powley's QPDs cannot handle many vectors of coherent QPDs. Also, additional steps are needed to engineer the support of Keelin and Powley's QPDs, such as by use of transformations – and some of these transformations may yield distributions with undefined or infinite moments, which can be problematic in practice if this violates an expert's beliefs.

We close by suggesting a couple of natural extensions to our work. The first is to develop a simple method of fitting a J-QPD distribution to an over-specified vector of assessed points. The linear form of the parameters in the SQN system allow Keelin and Powley (2011) to represent the least-squares fit problem as a simple quadratic program in the four unknown parameters, giving the SQN an advantage over the J-QPD system in the case of an over-specified system. Perhaps it might be possible to develop a five-point version of an SPT, with percent points spaced so as to yield a tractable regression problem in solving for the three unknown parameters.

Second, it would be very useful to develop a method of handling dependence among uncertainties using the J-QPD system. One natural candidate is under the conditional assessment regime, in which we assess three separate triplets of low-base-high values for uncertainty Y, conditional on the low-base-high assessments for uncertainty X. A natural alternative is to develop a way of tying the J-QPD distributions to a correlation and copula assessment approach, similar to that developed by Clemen and Reilly (1999).

In sum, we hope that this paper gives researchers and practitioners a deeper appreciation of the usefulness and convenience of distribution systems that are parameterized by assessed percentiles, as well as a more enriched understanding of distribution flexibility and measuring its extent. In addition, we hope that implementation of the J-QPD system becomes pervasive among practitioners, along with thoughtful consideration as to the appropriateness of its use.

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Appendix A – Proof of Proposition 1 (proposition restated for convenience)

Proposition 1 (MF Property). Consider any compatible $\boldsymbol{\theta}_\alpha = (l, \mathbf{x}_\alpha, u) = (l, x_\alpha, x_{0.50}, x_{1-\alpha}, u)$. There exists a unique quantile function, Q, characterized by (7), that satisfies:

- $Q_B(0) = l$
- $Q_B(\alpha) = x_\alpha$

- $Q_B(0.5) = x_{0.50}$
- $Q_B(1-\alpha) = x_{1-\alpha}$
- $Q_B(1) = u$

Proof. Since we are given $\boldsymbol{\theta}_\alpha$ as compatible, by Definition 1, it follows that $l < x_\alpha < x_{0.50} < x_{1-\alpha} < u$. Since $\boldsymbol{\theta}_\alpha$ is chosen arbitrarily, it suffices to show that the functional representation, $Q(p)$, given in Equation (7) corresponds to a quantile function for the given $\boldsymbol{\theta}_\alpha$. The expression, $\lambda \cdot \sinh(\delta \cdot (\Phi^{-1}(p) + n \cdot c)) + \xi$, in (7) corresponds to the QF for a Johnson SU distribution (by definition), provided that $\lambda > 0$ and $\delta > 0$. Since $\Phi(x)$ is increasing over the real number line, it follows that $Q(p)$ is an increasing quantile function for the given $\boldsymbol{\theta}_\alpha$ if and only if $\lambda > 0$ and $\delta > 0$ in accordance with the parameter expression in (7). There are three cases to consider:

<u>**Case 1**: *n* = -1</u>.

Referencing the parameter expressions given in (7), we have:

$$n = -1 \Leftrightarrow L + H - 2B < 0 \Leftrightarrow H - B < B - L, \text{ and } \frac{H-L}{2(H-B)} > 1.$$

This implies that:

$$\delta = \left(\frac{1}{c}\right)\cosh^{-1}\left(\frac{H-L}{2(H-B)}\right) > 0.$$
$$\Rightarrow \lambda = \frac{H-L}{\sinh(2\delta c)} > 0.$$

<u>**Case 2**: *n* = 1</u>.

Referencing the parameter expressions given in (7), we have:

$$n = -1 \Leftrightarrow L + H - 2B > 0 \Leftrightarrow H - B > B - L, \text{ and } \frac{H-L}{2(B-L)} > 1.$$

This implies that:

$$\delta = \left(\frac{1}{c}\right)\cosh^{-1}\left(\frac{H-L}{2(B-L)}\right) > 0.$$
$$\Rightarrow \lambda = \frac{H-L}{\sinh(2\delta c)} > 0.$$

<u>**Case 3**: *n* = 0</u>.

For the case in which $n = 0$, $Q(p)$ is defined as in (7b):

$$Q(p) = l + (u - l) \cdot \Phi\left(B + \left(\frac{H - L}{2c}\right) \cdot \Phi^{-1}(p)\right).$$

By inspection, *Q(p)* is non-decreasing as defined since $(u - l) > 0$, and since $H - L > 0$.

This completes the proof of Proposition 1. ■

Appendix B – Showing that $Q_1(p)$ (in §4.3) has Infinite Moments

Consider the following quantile function representation:

$$Q(p) = l + \theta \exp\left(\lambda \sinh(\delta(\Phi^{-1}(p) + \gamma))\right),\ \delta > 0,\ \lambda > 0,\ \theta > 0.$$

We seek to show that all positive moments of $Q(p)$ are infinite. Without loss of generality, we set $l = 0$ and $\theta = 1$, since these correspond to location and scale parameters, respectively. Let μ_k denote the k^{th} raw moment of $Q(p)$, for any $k > 0$. By definition, the quantile representation of μ_k is given by:

$$\mu_k = \int_0^1 Q^k(p)dp = \int_0^1 \exp\left(k \cdot \lambda \cdot \sinh\left(\delta \cdot (\Phi^{-1}(p) + \gamma)\right)\right)dp = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} + k \cdot \lambda \cdot \sinh\left(\delta \cdot (x + \gamma)\right)\right)dx.$$

The integrand is non-negative, and we note that:

$$\lim_{x\to\infty}\left(\exp\left(-\frac{x^2}{2} + k \cdot \lambda \cdot \sinh\left(\delta \cdot (x + \gamma)\right)\right)\right) = \infty,$$

since $k\lambda > 0$ and $\delta > 0$. Thus, the integral diverges, which implies that μ_k is infinite for $Q(p)$ whenever $k > 0$.

Appendix C – Finiteness of J-QPD-S Moments

As defined in Equation (9), the quantile function for the J-QPD-S distribution is given by:

$$Q_S(p) = l + \theta \cdot \exp\left(\lambda \cdot \sinh\left(\sinh^{-1}\left(\delta \cdot \Phi^{-1}(p)\right) + \sinh^{-1}(n \cdot c \cdot \delta)\right)\right).$$

We seek to show that all positive moments of $Q_S(p)$ are finite. There are three cases to consider: $n = \{-1, 0, 1\}$. For $n = 0$, recall that we recover a lognormal distribution, which is well-known to have finite positive moments. We now consider $n = \{-1, 1\}$. Without loss of generality, we remove location and scale. Consider:

$$Q_S(p) = \exp\left(\lambda \cdot \sinh\left(\sinh^{-1}\left(\delta \cdot \Phi^{-1}(p)\right) + \sinh^{-1}(n \cdot c \cdot \delta)\right)\right),\ \lambda > 0,\ \delta > 0.$$

Let μ_k denote the k^{th} raw moment associated with $Q_S(p)$, for any $k > 0$. By definition, the quantile representation of μ_k is given by:

$$\mu_k = \int_0^1 (Q_S(p))^k dp = \int_0^1 \exp\left(k \cdot \lambda \cdot \sinh\left(\sinh^{-1}\left(\delta \cdot \Phi^{-1}(p)\right) + \sinh^{-1}(n \cdot c \cdot \delta)\right)\right)dp$$

$$\int_0^1 \exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}+n\cdot c\cdot\delta\cdot\sqrt{1+\left(\delta\cdot\Phi^{-1}(p)\right)^2}\right)\right)dp.$$

Now, let us first consider *n* = -1.

<u>Case 1</u>: <u>*n* = -1</u>.

We have:

$$\mu_k=\int_0^1 \exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}-\underbrace{c\cdot\delta\cdot\sqrt{1+\left(\delta\cdot\Phi^{-1}(p)\right)^2}}_{>0}\right)\right)dp\le\int_0^1 \exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}-0\right)\right)dp$$

$$=\int_0^1 \exp\left(k\cdot\lambda\cdot\delta\cdot\sqrt{1+(c\cdot\delta)^2}\cdot\Phi^{-1}(p)\right)dp=\exp\left(\left(\frac{1}{2}\right)\cdot k^2\cdot\lambda^2\cdot\delta^2\cdot\left(1+(c\cdot\delta)^2\right)\right)<\infty. \quad \blacksquare$$

<u>Case 2</u>: <u>*n* = 1</u>.

In this case, we have:

$$\left(Q_S(p)\right)^k=\exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}+c\cdot\delta\cdot\sqrt{1+\left(\delta\cdot\Phi^{-1}(p)\right)^2}\right)\right).$$

We note the following:

$$\mu_k=\int_0^1\left(Q_S(p)\right)^k dp=\int_0^{1/2}\left(Q_S(p)\right)^k dp+\int_{1/2}^1\left(Q_S(p)\right)^k dp\le\int_0^{1/2}\left(Q_S(0.5)\right)^k dp+\int_{1/2}^1\left(Q_S(p)\right)^k dp$$

$$=\left(\frac{1}{2}\right)\exp\left(k\cdot\lambda\cdot c\cdot\delta\right)+\int_{1/2}^1\exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}+c\cdot\delta\cdot\sqrt{1+\left(\delta\cdot\Phi^{-1}(p)\right)^2}\right)\right)dp$$

$$\le\left(\frac{1}{2}\right)\exp\left(k\cdot\lambda\cdot c\cdot\delta\right)+\int_{1/2}^1\exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}+c\cdot\delta\cdot\left(1+\delta\cdot\Phi^{-1}(p)\right)\right)\right)dp$$

$$\le\left(\frac{1}{2}\right)\exp\left(k\cdot\lambda\cdot c\cdot\delta\right)+\int_0^1\exp\left(k\cdot\lambda\cdot\left(\delta\cdot\Phi^{-1}(p)\sqrt{1+(c\cdot\delta)^2}+c\cdot\delta\cdot\left(1+\delta\cdot\Phi^{-1}(p)\right)\right)\right)dp$$

$$=\left(\frac{1}{2}\right)\exp\left(k\cdot\lambda\cdot c\cdot\delta\right)+\exp\left(k\cdot\lambda\cdot\delta\cdot\left(c+\left(\frac{1}{2}\right)k\cdot\lambda\cdot\delta\cdot\left(c\cdot\delta+\sqrt{1+(c\cdot\delta)^2}\right)^2\right)\right)<\infty. \quad \blacksquare$$

Appendix D – Proof of Proposition 3 (proposition restated for convenience)

Proposition 2 (MF Property). Consider any compatible $\boldsymbol{\theta}_\alpha=(l,\mathbf{x}_\alpha,\infty)$. There exists a unique quantile function, *Q*, characterized by (9), that satisfies:

- $Q_S(0)=l$
- $Q_S(\alpha)=x_\alpha$
- $Q_S(0.5)=x_{0.50}$
- $Q_S(1-\alpha)=x_{1-\alpha}$.

- $Q_S(1) = \infty.$

Proof. Since we are given $\boldsymbol{\theta}_\alpha$ as compatible, by Definition 1, it follows that $l < x_\alpha < x_{0.50} < x_{1-\alpha} < \infty$. Since $\boldsymbol{\theta}_\alpha$ is chosen arbitrarily, it suffices to show that the functional representation, $Q_S(p)$, given in Equation (9) corresponds to a quantile function for the given $\boldsymbol{\theta}_\alpha$. Since $\Phi^{-1}(p)$ is increasing over *p*, and since the "Exp", "sinh", and "sinh^{-1}" operators are all non-decreasing over the real number line, it follows that $Q_S(p)$ is an increasing quantile function for the given $\boldsymbol{\theta}_\alpha$ if and only if $\theta > 0$, $\lambda > 0$ and $\delta > 0$ in accordance with the parameter expression in (9). There are three cases to consider:

Case 1: $n = -1$.

Referencing the parameter expressions given in (9), we first note that $\theta = x_{1-\alpha} - l > 0$, due to the given compatibility of $\boldsymbol{\theta}_\alpha$. Next, we observe that:

$$n = -1 \Leftrightarrow L + H - 2B < 0 \Leftrightarrow H - B < B - L, \text{ and } \frac{H-L}{2(H-B)} > 1.$$

This implies that:

$$\delta = \left(\frac{1}{c}\right)\sinh\left(\cosh^{-1}\left(\frac{H-L}{2(H-B)}\right)\right) > 0.$$
$$\Rightarrow \lambda = \left(\frac{1}{\delta \cdot c}\right)\cdot(H-B) > 0.$$

Case 2: $n = 1$.

Referencing the parameter expressions given in Equation (9), we first note that $\theta = x_\alpha - l > 0$, due to the given compatibility of $\boldsymbol{\theta}_\alpha$. Next, we observe that:

$$n = -1 \Leftrightarrow L + H - 2B > 0 \Leftrightarrow H - B > B - L, \text{ and } \frac{H-L}{2(B-L)} > 1.$$

This implies that:

$$\delta = \left(\frac{1}{c}\right)\sinh\left(\cosh^{-1}\left(\frac{H-L}{2(B-L)}\right)\right) > 0.$$
$$\Rightarrow \lambda = \left(\frac{1}{\delta \cdot c}\right)\cdot(B-L) > 0.$$

Case 3: $n = 0$.

For the case in which $n = 0$, $Q_S(p)$ is given as in Equation (9b):

$$Q_S(p) = l + \theta\exp\left(\lambda\delta\Phi^{-1}(p)\right), \text{ with}$$
$$\theta = x_{0.5} - l > 0,$$

$$\lambda\delta = \frac{H-B}{c} = \frac{B-L}{c} > 0.$$

Thus, $Q_S(p)$ is non-decreasing over p in this case, and is thus a quantile function. This completes the proof of Proposition 3. ■