1. The J-QPD-S (Semi-Bounded) Distributions

Let l denote a specified finite lower bound of support, and let $\{x_\alpha, x_{0.5}, x_{1-\alpha}\}$ denote a specified symmetric percentile triplet for some given $\alpha \in (0, 0.5)$. For example, $\{x_{0.1}, x_{0.5}, x_{0.9}\}$ corresponds to the {10th, 50th, 90th} percentiles. Finally, let ϕ, Φ, and Φ^{-1} denote the PDF, CDF, and quantile function for the standard normal distribution, respectively. The quantile function, CDF, and PDF for the J-QPD-S distribution parameterized by $\boldsymbol{\theta}_\alpha = (l, x_\alpha, x_{0.5}, x_{1-\alpha})$ are given as follows:

1.1 Standard Parameterizations

Quantile Function

$$Q_S(p) = l + \theta \exp\left(\lambda \sinh\left(\sinh^{-1}\left(\delta \Phi^{-1}(p)\right) + \sinh^{-1}(nc\delta)\right)\right).$$

Cumulative Distribution Function (CDF)

$$F_S(x) = \Phi\left(\left(\frac{1}{\delta}\right)\sinh\left(\sinh^{-1}\left(\left(\frac{1}{\lambda}\right)\log\left(\frac{x-l}{\theta}\right)\right) - \sinh^{-1}\left(nc\delta\right)\right)\right).$$

Probability Density Function (PDF)

$$f_S(x) = \left(\frac{1}{\lambda\delta(x-l)}\right)\cdot\left(k - nc\delta\cdot\left(\frac{\log\left(\frac{x-l}{\theta}\right)}{\sqrt{\lambda^2 + \log^2\left(\frac{x-l}{\theta}\right)}}\right)\right)\cdot\phi\left(\left(\frac{1}{\lambda\delta}\right)\cdot\left(k\log\left(\frac{x-l}{\theta}\right) - nc\delta\sqrt{\lambda^2 + \log^2\left(\frac{x-l}{\theta}\right)}\right)\right).$$

where,

$$c = \Phi^{-1}(1-\alpha),$$
$$L = \log(x_\alpha - l),\ B = \log(x_{0.5} - l),\ H = \log(x_{1-\alpha} - l),$$
$$n = \operatorname{sgn}(L + H - 2B),$$
$$\theta = \begin{cases} x_\alpha - l, & n = 1 \\ x_{0.50} - l, & n = 0 \\ x_{1-\alpha} - l, & n = -1 \end{cases},$$
$$\delta = \left(\frac{1}{c}\right)\sinh\left(\cosh^{-1}\left(\frac{H-L}{2\min(B-L, H-B)}\right)\right),$$
$$\lambda = \left(\frac{1}{\delta c}\right)\min(H-B, B-L),$$
$$k = \sqrt{1 + (c\delta)^2}.$$

1.2 Alternative ("Expanded") Parameterizations

Alternatively, "expanding out" the hyperbolic functions in the expressions above gives the following alternative forms for the J-QPD-S quantile function and CDF[1]:

Quantile Function

$$Q_S(p) = l + \theta \cdot \exp\left(\lambda\delta\left(k\Phi^{-1}(p) + nc\sqrt{1+\left(\delta\cdot\Phi^{-1}(p)\right)^2}\right)\right).$$

Cumulative Distribution Function (CDF)

$$F_S(x) = \Phi\left(\left(\frac{1}{\lambda\delta}\right)\cdot\left(k\log\left(\frac{x-l}{\theta}\right) - nc\delta\sqrt{\lambda^2 + \log^2\left(\frac{x-l}{\theta}\right)}\right)\right).$$

1.3 Special Case

The lognormal distributions occur as a special case of the J-QPD-S distributions. Specifically, for the special case in which $L+H-2B=0$, we have $n = \text{sgn}(L+H-2B) = \text{sgn}(0) = 0$, and thus:

$$Q_S(p) = l + \theta\exp\left(\lambda\sinh\left(\sinh^{-1}\left(\delta\Phi^{-1}(p)\right)\right)\right) = l + \theta\cdot\exp\left(\lambda\delta\Phi^{-1}(p)\right),$$

$$F_S(x) = \Phi\left(\left(\frac{1}{\lambda\delta}\right)\cdot\log\left(\frac{x-l}{\theta}\right)\right),$$

$$f_S(x) = \left(\frac{1}{\lambda\delta(x-l)}\right)\cdot\phi\left(\left(\frac{1}{\lambda\delta}\right)\cdot\log\left(\frac{x-l}{\theta}\right)\right).$$

This corresponds to a lognormal distribution with parameters, $\mu = \log(\theta)$ and $\sigma = \lambda\delta$, and shifted to have support on (l, ∞).

2. The J-QPD-B (Bounded) Distributions

Let l and u denote specified finite lower and upper bounds of support, respectively, and let $\{x_\alpha, x_{0.5}, x_{1-\alpha}\}$ denote a specified symmetric percentile triplet for some given $\alpha \in (0, 0.5)$. For example, $\{x_{0.1}, x_{0.5}, x_{0.9}\}$ corresponds to the {10^{th}, 50^{th}, 90^{th}} percentiles. Finally, let ϕ, Φ, and Φ^{-1} denote the PDF, CDF, and quantile function for the standard normal distribution, respectively. The quantile function, CDF, and PDF for the J-QPD-B distribution parameterized by $\boldsymbol{\theta}_\alpha = (l, x_\alpha, x_{0.5}, x_{1-\alpha}, u)$ are given as follows:

Quantile Function

$$Q_B(p) = l + (u-l)\Phi\left(\xi + \lambda\sinh\left(\delta\left(\Phi^{-1}(p) + nc\right)\right)\right).$$

Cumulative Distribution Function (CDF)

$$F_B(x) = \Phi\left(\left(\frac{1}{\delta}\right)\sinh^{-1}\left(\left(\frac{1}{\lambda}\right)\left(\Phi^{-1}\left(\frac{x-l}{u-l}\right) - \xi\right)\right) - nc\right).$$

[1] The PDF is already in expanded form.

Probability Density Function (PDF)

$$f_B(x)=\frac{\phi\left(-nc+\left(\frac{1}{\delta}\right)\sinh^{-1}\left(\left(\frac{1}{\lambda}\right)\left(-\xi+\Phi^{-1}\left(\frac{x-l}{u-l}\right)\right)\right)\right)}{\delta(u-l)\sqrt{\lambda^2+\left(-\xi+\Phi^{-1}\left(\frac{x-l}{u-l}\right)\right)^2}\phi\left(\Phi^{-1}\left(\frac{x-l}{u-l}\right)\right)}$$

where,

$$c=\Phi^{-1}(1-\alpha)\ ,$$
$$L=\Phi^{-1}\left(\frac{x_\alpha-l}{u-l}\right),\ B=\Phi^{-1}\left(\frac{x_{0.50}-l}{u-l}\right),\ H=\Phi^{-1}\left(\frac{x_{1-\alpha}-l}{u-l}\right),$$
$$n=\operatorname{sgn}(L+H-2B),$$
$$\xi=\begin{cases} L,\ n=1\\ B,\ n=0\\ H,\ n=-1\end{cases}$$
$$\delta=\left(\frac{1}{c}\right)\cosh^{-1}\left(\frac{H-L}{2\min(B-L,H-B)}\right),$$
$$\lambda=\frac{H-L}{\sinh(2\delta c)}.$$

2.1 Special Case

For the special case in which n = 0, the quantile function, CDF, and PDF for J-QPD-B are given by:

$$Q_B(p)=l+(u-l)\Phi\left(B+\left(\frac{H-L}{2c}\right)\Phi^{-1}(p)\right),$$
$$F_B(x)=\Phi\left(\left(\frac{2c}{H-L}\right)\cdot\left(-B+\Phi^{-1}\left(\frac{x-l}{u-l}\right)\right)\right),$$
$$f_B(x)=\frac{2c\cdot\phi\left(\left(\frac{2c}{H-L}\right)\cdot\left(-B+\Phi^{-1}\left(\frac{x-l}{u-l}\right)\right)\right)}{(H-L)(u-l)\cdot\phi\left(\Phi^{-1}\left(\frac{x-l}{u-l}\right)\right)}.$$